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PHILOSOPHICAL TRANSACTIONS.

XI. *An essay towards the calculus of functions. Part* II. *By* C. Babbage, *Esq. Communicated by* W. H. Wollaston, *M. D. Sec. R. S.*

Read March 14, 1816.

In a former Paper which the Royal Society honoured with a place in the last Volume of their Transactions, I endeavoured to explain the nature of the calculus of functions, and I proposed means of solving a variety of functional equations containing only one variable quantity. My subsequent enquiries have produced several new methods of solving these, and much more complicated functional equations, and have convinced me of the importance of the calculus, particularly as an instrument of discovery in the more difficult branches of analysis; nor is it only in the recesses of this abstract science, that its advantages will be felt: it is peculiarly adapted to the discovery of those laws of action by which one particle of matter attracts or repels another of the same or of a different species; consequently, it may be applied to every branch of natural philosophy, where the object is to discover by calculation from the results of experiment, the laws which

regulate the action of the ultimate particles of bodies. To the accomplishment of these desirable purposes, it must be confessed that it is in its present state unequal; but should the labours of future enquirers give to it that perfection, which other methods of investigation have attained, it is not too much to hope, that its maturer age shall unveil the hidden laws which govern the phenomena of magnetic, electric, or even of chemical action.

When functional equations containing two or more variables occur, their solution presents still greater difficulties than those we have already considered; the new relations which arise, necessarily require a new notation to distinguish them. I shall endeavour, as far as I am able, to apply or extend that already in use; but, as it is almost impossible in the infancy of a calculus to foresee the extent to which it may be carried, or the new views which it may be necessary to take of it, the notation I have used should only be considered as of a temporary nature; it may be employed until some more convenient one be devised: perhaps, however, it might be more advantageous that it should not be altered until our acquaintance with this subject becomes more intimate, and until the infinitely varied and comprehensive relations displayed in the doctrine of functions, have been more minutely examined.

If ψ be the characteristic of any function of two quantities x and y, that function is thus denoted $\psi(x, y)$. Now, if instead of x in this quantity the original function be substituted, I shall call the result the second function relative to x, and I shall denote it thus

$$\psi^{2,1}(x, y) = \psi(\psi(x, y), y)$$

The first index 2 refers to x, and the second index 1 refers to y. Similarly if instead of y in the original function, the function itself had been substituted, the result would have been the second function relative to y; it would be thus denoted

$$\psi^{1,2}(x,y) = \psi(x, \psi(x,y))$$

If there are more than two variables in the original function, they may be arranged in the order in which they are to be operated on, and the indices will denote the number of operations to be performed.

Thus $\psi^{2,3,1,1}(x, y, z, v,)$ signifies that in the function $\psi(x,y,z,v,)$ we must instead of x substitute the function itself, and in the result instead of y put the same function, this latter operation must be repeated, and finally, instead of v in the last result, put the original function; this last operation must again be repeated twice.

There are many cases which this notation does not comprehend. If, for example, in the function just proposed, we wished again to take the function relative to x or y, it would not be easy to express this. The method I propose is to have two ranks of indices, the lower one to distinguish the quantities operated on; the upper one to mark the number of operations performed. According to this method the example just chosen would be written thus:

$$\psi^{\substack{2,3,1,4\\1,3,3,4}}(x,y,z,v)$$

If only such functions as these occur, we encumber our symbol without any advantage; if, however, we now wish to perform any farther operations, such, for instance, as to take the second function relative to z, and then the third relative to y,

we have a very convenient mode of doing it; these operations would be thus expressed;

$$\psi^{\substack{2,3,1,4,2,3\\1,2,3,4,3,2}}(x,y,z,v)$$

This notation may not appear sufficiently concise to those who do not consider the very complicated relation expressed by the above written symbol: it need, however, only be used in very few cases, and when the lower series of indices is omitted, it must always be understood, that the quantities themselves are arranged in the order in which they are to be operated on.

If in a function of two variable $\psi(x,y,)$ we take the second function relative to x, and then the second function relative to y, we have

$$\psi^{\substack{2,2\\1,3}}(x,y) = \psi\left\{\psi\left(x,\psi(x,y)\right)\ \psi(x,y)\right\}$$

If we take the second function first relative to y, and then the second relative to x we shall find

$$\psi^{\substack{2,2\\2,1}}(x,y) = \psi\left\{\psi(x,y),\psi(\psi(x,y),y)\right\}$$

It appears from this, that the order in which these operations are performed is not immaterial, as the order in which we differentiate a function of two variables, is in the differential calculus.

The two expressions just given are the two second functions of $\psi(x,y)$, the first taken relative to x and y, and the second taken relative to y and x. But there may be another second function of $\psi(x,y)$, which will arise from substituting at the same time $\psi(x,y)$ for x, and $\psi(x,y)$ for y, it will be

$$\psi(\psi(x,y),\psi(x,y))$$

and may for the sake of distinction be called the second simultaneous function relative to x and y; it differs from the two preceding ones, and in order to denote it with brevity, I shall put a line over the two indices thus,

$$\overline{\psi^{2,2}}(x,y) = \psi(\psi(x,y), \psi(x,y))$$

This method of distinguishing it is equally applicable when there are more variables.

There is only one other modification of the symbol denoting function to which I shall at present allude. Suppose (after any number of operations have been performed on a function of two variables for example) y becomes equal to x, and the result only is given: this will naturally be represented in a manner analogous to that in which EULER denoted the limits between which the integral of a quantity is to be taken.

Thus the equation $\psi^{2,2}_{1,2}(x,y) = fx \quad [y=x]$ arises from the following question: What is the form of a function of x and y, such that taking the second function relative to x, and then the second relative to y, the result on making y equal to x shall be a given function of x?

It might be proposed, that after putting y equal to x, the whole should be considered merely as a function of x, and that its n^{th} function should be taken on this hypothesis, and the result only should be given.

Such operations I would denote thus:

$$\left\{\psi^{2,2}_{1,2}\right\}^n(x,y) = f(x)\,[y=x] \text{ or perhaps } \left\{\psi^{2,2}_{1,2}\right\}^n(x,y) = f(x)[y=x]$$

and in a similar manner all other relations of the same kind may be expressed.

I shall give one example which will illustrate these various modifications of the original functions,

$$\overset{2,\,3,\,\overline{2,\,2}\}\,3,\,2\}^{3}}{\psi\,(x,y,z,v)} = f(x) \quad \left[\begin{matrix} y = \alpha x \\ v = \beta z \end{matrix} \quad z = \gamma x\right]$$

This equation contains the analytical enunciation of the following Problem.

What must be the form of a function of four quantities $\psi(x,y,z,v)$ such that taking the second function relative to z, the third relative to x, and the second simultaneous one relative to y and v: if in the result αx be put for y and βz for v, and the whole be then considered as a function of x and z, and if on this hypothesis the third function be taken relative to z, and the second relative to x; and if γx be now put for z and the third function of the expression considered merely as a function of x be taken, then it is required that the final result shall be equal to fx a given function of x?

Symmetrical functions I shall denote as in my former Paper, by putting a line over the quantities relative to which they are symmetrical, thus $\psi(\overline{x,y},\overline{z,v})$ is symmetrical relative to x and y in one sense, and relative to z and v in another.

PROBLEM I.

Required the solution of the functional equation

$$\psi(x,y) = \psi(\alpha x, \beta y)$$

To avoid repetition α, β, γ, &c. unless otherwise mentioned, always express known functions, and φ, ψ, χ are unknown or arbitrary ones.

Put $\psi(x,y,) = \varphi(fx,fy,)$

then the given equation becomes

$$\phi(fx, {}_1fy) = \phi(f_0 x, {}_1f\beta y)$$

Determine f and ${}_1f$ from the two equations

$$fx = f_0 x \text{ and } {}_1fy = {}_1f\beta y$$

this may be effected by Prob. I. or II. of my former Paper, take any particular solution and ϕ may remain perfectly arbitrary; then the general solution of the problem is

$$\psi(x, y) = \phi(fx, {}_1fy)$$

Ex. 1. Given the equation $\psi(x, y) = \psi\left(-x, \frac{1}{y}\right)$ here we have $f(x) = f(-x)$, and a particular solution is $fx = x^2$; also $f(y) = {}_1f\left(\frac{1}{y}\right)$ and a particular case is ${}_1f(y) = \frac{y^2+1}{y}$ hence the general solution of the equation is

$$\psi(x, y) = \phi\left(x^2, \frac{y^2+1}{y}\right)$$

ϕ being perfectly arbitrary.

If we employ the general solutions of the equations $f(x) = f(-x)$ and $f(y) = {}_1f\left(\frac{1}{y}\right)$, we shall still only have one arbitrary function. In fact, the most general solution of the equation $\psi(x, y) = \psi\left(-x, \frac{1}{y}\right)$ with which I am at present acquainted is

$$\psi(x, y) = \phi\left\{\begin{matrix} - & - & \frac{1}{} & \frac{1}{} \\ -x, & x, & y, & \frac{1}{y} \end{matrix}\right\}$$

and this only involves one arbitrary function.

PROBLEM II.

Given the same equation

$$\psi(x,y) = \psi(\alpha x, \beta y)$$

Suppose one particular solution of this equation is known, let it be $f(x,y)$,

then take $\psi(x,y) = \varphi f(x,y)$, φ being perfectly arbitrary and the given equation becomes

$$\varphi f(x,y) = \varphi f(\alpha x, \beta y)$$

which is evidently satisfied since $f(x,y) = f(\alpha x, \beta y)$ by the hypothesis.

Ex. 1. Let $\psi(x,y) = \psi\left(x^n, y^{\frac{1}{n}}\right)$

one particular solution of this equation is $f(x,y) = x^{\log. y}$

hence the general solution is

$$\psi(x,y) = \varphi(x^{\log. y})$$

Ex. 2. Given the equation $\psi(x,y) = \psi(x^n, y^n)$ a particular case is $f(x,y) = \frac{\log. x}{\log. y}$, hence the general solution is

$$\psi(x,y) = \varphi\left(\frac{\log. x}{\log. y}\right)$$

Ex. 3. Given the equation $\psi(x,y) = \psi(x^n, y^m)$

In order to get a particular case let us put

$$f(x,y) = \log.^n x + a \log.^n y$$

by substituting this value we shall find that it is a particular solution of the equation, if $a = -\frac{\log. n}{\log. m}$,

hence the general solution of the equation is

$$\psi(x,y) = \varphi\left(\log.^n x - \frac{\log. n}{\log. m}\log.^n y\right) = \varphi\left(\log.\frac{\log. x}{(\log. y)^{\frac{\log. n}{\log. m}}}\right)$$

or changing the value of φ it becomes

$$\psi(x,y) = \varphi\left(\frac{(\log. x)^{\log. m}}{(\log. y)^{\log. n}}\right)$$

If $n = m$ we have $\psi(x,y) = \varphi\left(\frac{\log. x}{\log. y}\right)$ as in the last example, and if $m = \frac{1}{n}$, we have the same solution as in the first.

In these equations the functions have contained the variables separated; but it may frequently happen, that they occur mixed as in the following Problems.

PROBLEM III.

Given the equation

$$\psi(x,y) = \psi(\alpha(x,y), \beta(x,y))$$

Assume $\psi(x,y) = \varphi(f(x,y), f_1(x,y))$, and by making this substitution the equation becomes

$$\varphi\left\{f(x,y), f_1(x,y)\right\} = \varphi\left\{f(\alpha(x,y),\beta(x,y)), f_1(\alpha(x,y),\beta(x,y))\right\}$$

In order to render this equation identical, I determine f and f_1 from the two following equations:

$$f(x,y) = f(\alpha(x,y), \beta(x,y)) \text{ and } f_1(x,y) = f_1(\alpha(x,y),\beta(x,y))$$

From these it appears, that f and f_1 are merely two particular solutions of the original equation. If, therefore, we are acquainted with any, the general solution is

$$\psi(x,y) = \varphi(f(x,y), f_1(x,y))$$

If only one particular solution is known, the general one is

$$\psi(x,y) = \varphi f(x,y)$$

Ex. 1. Let us examine in what cases we can find the general solution of the equation

$$\psi(x,y) = \psi(x^n y^m, x^r y^s)$$

In order to obtain a particular solution, put $\psi(x,y) = x^\nu y^\mu$ and making this substitution, we shall find the following equation of condition among the exponents.

$$(1-n)(1-r) = km$$

hence either of the following equations may be solved generally,

$$\psi(x,y) = \psi\left\{\frac{x^r}{y^{r-1}}, \frac{y^r}{x^{n-1}}\right\} \qquad (a)$$

$$\psi(x,y) = \psi\left\{y\left(\frac{x}{y}\right)^n, x\left(\frac{y}{x}\right)^r\right\} \qquad (b)$$

the solution of the former is $\psi(x,y) = \varphi(xy)$, and that of the latter is $\psi(x,y) = \varphi(x^{r-1}y^{n-1})$

In (a) put $n = r = \frac{1}{2}$, then the solution of $\psi(x,y) = \left(\sqrt{xy}, \sqrt{xy}\right)$ is $\psi(x,y) = \varphi(xy)$

In a similar manner it may be found that the solution of

$$\psi(x,y) = \psi\left(\frac{x-y}{y}, \frac{x-y}{x}\right)$$

is $$\psi(x,y) = \varphi\left(\frac{y}{x}\right)$$

As another example take the equation

$$\psi(x,y) = \psi\left(\frac{y}{2}\sqrt{\frac{y}{2x}}, \sqrt{2xy}\right)$$

a particular solution is $\psi(x,y) = 2xy + y^2$, hence the general solution is $\psi(x,y) = \varphi(2xy + y^2)$, but we may find another particular solution of this equation which is totally different from the former, and by combining the two we shall obtain a much more general solution. The equation (b) will coincide with the one under consideration, if we make $n = -\frac{1}{2}$ and $r = \frac{1}{2}$, then we shall have for another particular solution $f(x,y) = \frac{1}{y\sqrt{2xy}}$, hence a very general solution of the equation $\psi(x,y) = \psi\left(\frac{y}{2}\sqrt{\frac{y}{2x}}, \sqrt{2xy}\right)$ is

$$\psi(x,y) = \varphi\left\{2xy + y^2, \frac{1}{y\sqrt{2xy}}\right\}$$

φ remaining perfectly arbitrary.

In the equation of this Problem it may happen that

$\alpha(x,y)$ does not contain x nor $\beta(x,y)$ contain y, it then becomes $\psi(x,y) = \psi(\alpha y, \beta x)$

This is the case when $\psi(x,y)$ is required to be a symmetrical function of x and y, the equation would then become

$$\psi(x,y) = \psi(y,x)$$

two particular solutions are $f(x,y) = xy$ and $f(x,y) = x+y$, hence the general solution of the equation is

$$\psi(x,y) = \varphi(xy, x+y)$$

Though these solutions may with propriety be termed general because they contain an arbitrary function, yet I am by no means inclined to think them the most general of which the questions admit, possibly we ought to except the two last equations, though I shall afterwards show that the solution of an equation of the form $\psi(x,y) = \psi(\alpha x, \beta y)$ may contain any number of known functions within the arbitrary one.

Problem IV.

Given the equation

$$\psi(x,y) = \psi(\alpha(x,y), \beta(x,y))$$

Assume as before $\psi(x,y) = \varphi(f(x,y), f_1(x,y))$, then the equation will become

$$\varphi(f(x,y), f_1(x,y)) = \varphi\left\{f(\alpha(x,y), \beta(x,y)), f_1(\alpha(x,y), \beta(x,y))\right\}$$

In order to render this equation identical, determine f and f_1 from the two equations

$$f(x,y) = f(\alpha(x,y), \beta(x,y)) \text{ and } f_1(x,y) = f(\alpha(x,y), \beta(x,y))$$

putting in the first of these $\alpha(x,y)$ for x and $\beta(x,y)$ for y we find

$$f(\alpha(x,y), \beta(x,y)) = f\left\{\alpha(\alpha(x,y), \beta(x,y)), \beta(\alpha(x,y), \beta(x,y))\right\}$$
$$= f_1(x,y) \qquad (1)$$

and we should find a precisely similar equation for determining f. If we are acquainted with two particular solutions of this equation, we may from them derive the general solution of the given equation. If, however, the functions α and β are of such a nature that the two following equations are fulfilled eq. (1) becomes identical without assigning any particular value to f or f. (The two conditions are $\alpha(\alpha(x,y),\beta(x,y))=x$ and $\beta(\alpha(x,y),\beta(x,y))=y$).

It may be curious to enquire whether we can discover any forms which will satisfy these equations, for this purpose let us assume $\alpha(x,y)=a+bx+cy$, and also $\beta(x,y)=a+bx+cy$, this will only lead us to a particular solution, but I shall presently show that it may be rendered general. If the two conditions already specified are fulfilled, the arbitrary constants will be determined, and we shall have the following equations

$$\alpha(x,y)=a+bx+\frac{b^n-1}{b}y$$

$$\beta(x,y)=\frac{ab}{1-b}-bx-by$$

which may be thus generalised. Let ϕ be any function, and let $\overline{\phi}^1$ be the inverse of that function, so that $\phi\overline{\phi}x=x$ then the conditions will be fulfilled, if

$$\alpha(x,y)=\overline{\phi}^1\left\{a+b\phi x+\frac{b^n-1}{b}\phi y\right\}$$

and $$\beta(x,y)=\overline{\phi}^1\left\{\frac{ab}{1-b}-b\phi x-b\phi y\right\}$$

Some remarks, however, are necessary on the inverse function $\overline{\phi}^1$. If we combine x and constant quantities by any of the direct operations, addition, multiplication, elevation of powers, &c. the result which is called a function of x admits

only of one value, let z equal the function, then we have the equation $z = \phi x$. If from this we endeavour to discover the value of x in terms of z, the operation is an inverse one and x admits of one or more values according to the nature of the operations denoted by ϕ. This number may even be infinite; if ϕ denotes an equation of the n^{th} degree, there are n values of x in terms of z. It may then be enquired whether in using the substitution employed in the latter part of this Problem, any of these (perhaps infinite number) may be taken, or whether only certain particular values should be used? without attending to this circumstance, our conclusions may become erroneous: all these different values will satisfy the equation $\phi\phi^{-1}x = x$, but only those must be used which also satisfy the equation $\phi^{-1}\phi x = x$: thus if $z = \phi x = a - x^2$ we shall have $x = \phi^{-1}z = \pm\sqrt{a-z}$ if we employ the upper sign we have

$$\phi\phi^{-1}x = +\sqrt{a-(a-x^2)} = +\sqrt{x^2} = +x$$

If we use the lower one

$$\phi^{-1}\phi x = -\sqrt{a-(a-x^2)} = -\sqrt{x^2} = -x$$

the upper sign must therefore be taken, because in the latter part of the Problem we suppose $\phi^{-1}\phi x = x$ and $\phi^{-1}\phi y = y$. This remark, which is of some importance, extends to the conclusions in my former Paper and to the whole of the subsequent enquiries.

The equation (1) might be considered as similar to the original one, and the same transformation might be performed on this, and thus we might continue to deduce new conditions. In the first part we found that the equation $\psi x = \psi_n x$ always admitted of an easy solution when $\alpha^n x = x$ and by continuing the substitutions already pointed out, we should arrive at

some conclusions very analogous for functional equations of the form of those treated of in this Problem, but the length to which these enquiries would lead, render it sufficient merely to indicate them.

In the equations solved in Problem I. and II. it is obviously immaterial whether we first put αx instead of x, and then in the result put βy for y or conversely; but in the equation of Problems III. and IV. the case is different. If in the function $\psi(x,y)$ we put simultaneously $\alpha(x,y)$ for x, and $\beta(x,y)$ for y the result will be different from that which would arise from first putting $\alpha(x,y)$ for x and then in the result putting $\beta(x,y)$ for y, or from inverting this operation; the three results stand thus:

$$\psi(\alpha(x,y),\beta(x,y)) \qquad (a)$$
$$\psi(\alpha(x,\beta(x,y)),\beta(x,y)) \qquad (b)$$
$$\psi(\alpha(x,y),\beta(\alpha(x,y),y)) \qquad (c)$$

These three functions are evidently different, and in the solutions of the Problems, regard was only had to the first of them which may be called the simultaneous function. Those, however, of the second and third class might occur, and it becomes necessary to point out the means of solution which are applicable to them.

According to the notation laid down, these functions may be thus expressed

$$\overset{1,1}{\psi}(\alpha(x,y),\beta(x,y)) \qquad (a)$$
$$\overset{1,2}{\psi}(\alpha(x,y),\beta(x,y)) \qquad (b)$$
$$\overset{2,1}{\psi}(\alpha(x,y),\beta(x,y)) \qquad (c)$$

But to avoid the trouble of indices I shall show how those of

the second and third class may be reduced to those of the first, I shall therefore always consider functions of the first order as simultaneous ones, and omit the indices, which if supplied, would be $\overline{{}^{1,\,1}_{1,\,2}}$, $\overline{{}^{1,\,1}_{1,\,3}}$, &c.

To transform $\psi^{1,\,2}(\alpha\,x,y),\ \beta\,(x,y))$ into a function whose index is ${}^{1,\,1}_{1,\,2}$ put $\alpha\,(x,\beta\,(x,y)) = \gamma\,(x,y)$ then

$$\overline{\psi^{1,\,2}}(\gamma\,(x,y),\beta\,(x,y)) = \psi^{2,\,1}(\alpha\,(x,y),\beta(x,y))$$

and similarly if $\beta\,(\alpha\,(x,y),\ \beta\,(x,y)) = \gamma\,(x,y)$ we should have

$$\psi^{2,\,1}(\alpha\,(x,y),\beta\,(x,y)) = \overline{\psi^{1,\,2}}(\alpha\,(x,y),\ \gamma\,(x,y))$$

and generally whatever be the number of variables a similar transformation might be effected.

PROBLEM V.

Required the solution of the equation,

$$\psi\,(x,y) = \mathrm{A}\,(x,y)\,\psi\,(\alpha\,(x,y),\beta\,(x,y))$$

Assume $\psi\,(x,y) = f\,(x,y)\,\varphi\left\{{}_1f\,(x,y),{}_2f\,(x,y)\right\}$ and substituting this in the given equation, we find

$$f(x,y)\,\varphi\left\{{}_1f\,(x,y),{}_2f\,(x,y)\right\} = \mathrm{A}\,(x,y)f\,(\alpha\,(x,y),\beta\,(x,y))$$
$$\times\,\varphi\left\{{}_1f\,(\alpha\,(x,y),\beta\,(x,y)),{}_2f\,(\alpha\,(x,y),\beta(x,y))\right\}$$

This equation will be satisfied if we are acquainted with particular solutions of the three following equations

$$f\,(x,y) = \mathrm{A}\,(x,y)f\,(\alpha\,(x,y),\beta\,(x,y))$$

${}_2f\,(x,y) = {}_1f\,(\alpha\,(x,y),\beta\,(x,y))$ and ${}_1f\,(x,y) = {}_1f\,(\alpha\,(x,y),\beta\,(x,y))$

the first of these is nothing more than the original equation.

If therefore we know one particular solution of the original equation, and also one or two particular solutions of the other equation, we may deduce the general solution of the Problem.

Ex. Let $\psi(x,y) = \left(\frac{x}{y}\right)^n \psi(y,x)$

in this case $f_1(x,y) = f_2(y,x)$ and two particular solutions are $f_1(x,y) = xy$ and $f_2(x,y) = x+y$ also a particular solution of the given equation is $f(x,y) = \frac{x}{y^n}$, hence its general solution is

$$\psi(x,y) = \frac{x}{y^n}\varphi(x+y, xy)$$

PROBLEM VI.

Given the equation

$$\psi(x,y) = A(x,y)\,\psi(\alpha(x,y), \beta(x,y)) + B(x,y)$$

Suppose we are acquainted with one particular solution which satisfies the equation and let it be $f(x,y)$, then assume

$$\psi(x,y) = f(x,y) + \varphi(x,y)$$

and making this substitution the equation becomes

$$f(x,y) + \varphi(x,y) = A(x,y) f(\alpha(x,y), \beta(x,y)) + A(x,y) \times \varphi(\alpha(x,y), \beta(x,y)) + B(x,y)$$

Substracting from this the particular solution

$$f(x,y) = A(x,y) f(\alpha(x,y), \beta(x,y)) + B(x,y)$$

there remains

$$\varphi(x,y) = A(x,y)\,\varphi(\alpha(x,y), \beta(x,y))$$

an equation which may be solved by the preceding Problem. The same substitution is applicable to the more general equation

$$0 = \psi(x,y) + A(x,y)\psi(\alpha(x,y),\beta(x,y)) + B(x,y)\psi(\alpha_1(x,y),\beta_1(x,y)) + \&c. + K(x,y)$$

PROBLEM VII.

Given the functions

$$\alpha(x,y), \underset{1}{\alpha}(x,y), \underset{2}{\alpha}(x,y) \quad \&c.$$

$$\beta(x,y), \underset{1}{\beta}(x,y), \underset{2}{\beta}(x,y) \quad \&c.$$

Required the nature of the function $\psi(x,y)$ such that it shall not alter its form by the simultaneous substitution of $\alpha(x,y), \beta(x,y)$ for x and y, and generally that it shall remain the same when for x and y are respectively substituted any of the functions denoted by $\underset{n}{\alpha}(x,y)$ and $\underset{n}{\beta}(x,y)$. The conditions which determine ψ may be thus expressed

$$\psi(x,y) = \psi(\alpha(x,y), \beta(x,y)) = \psi(\underset{1}{\alpha}(x,y), \underset{1}{\beta}(x,y)) = \&c.$$

Assume $$\psi(x,y) = \varphi\left\{f(x,y), \underset{1}{f}(x,y)\right\} \qquad (1)$$

then from the first condition we have

$$\varphi\left\{f(x,y), \underset{1}{f}(x,y)\right\} = \varphi\left\{f(\alpha(x,y), \beta(x,y)), \underset{1}{f}(\alpha(x,y), \beta(x,y))\right\}$$

this will be satisfied by making

$$f(x,y) = f(\alpha(x,y), \beta(x,y)) \text{ and } \underset{1}{f}(x,y) = \underset{1}{f}(\alpha(x,y), \beta(x,y))$$

these are two particular solutions of the first equation.

The second condition is $\psi(x,y) = \psi(\underset{1}{\alpha}(x,y), \underset{1}{\beta}(x,y))$

which becomes

$$\varphi(f(x,y), \underset{1}{f}(x,y)) = \varphi(f(\underset{1}{\alpha}(x,y), \underset{1}{\beta}(x,y)), \underset{1}{f}(\underset{1}{\alpha}(x,y), \underset{1}{\beta}(x,y))) \qquad (2)$$

where f and $\underset{1}{f}$ are known functions; make

$$f(\underset{1}{\alpha}(x,y), \underset{1}{\beta}(x,y)) = \mathrm{K}(x,y) \text{ and } \underset{1}{f}(\underset{1}{\alpha}(x,y), \underset{1}{\beta}(x,y)) = \underset{1}{\mathrm{K}}(x,y)$$

K and $\underset{1}{\mathrm{K}}$ are therefore also known functions.

Assume $\phi(x,y) = \phi_1(f_2(x,y), f_2(x,y))$, then equation (2) becomes

$$\phi_2\{f_2(f(x,y), f_1(x,y)), f_2(f(x,y), f_1(x',y))\} = \phi_2\{f_1(\mathrm{K}(x,y), \mathrm{K}(x,y)), f_1(\mathrm{K}(x,y)\,\mathrm{K}(x,y))\}$$

This equation must be solved in the same manner as the former by means of two particular solutions, and by continuing the same method, we shall find that the form of the function ψ may be determined by means of $2n$ particular solutions of certain functional equations, when there are n pair of conditions assigned. A less general solution may, however, be found when we are only acquainted with n particular solutions.

A similar method would lead us to the form of ψ, whatever might be the number of variables. If, however, we are acquainted with any number of particular solutions which remain the same, in all the cases assigned by the conditions of the Problem, we may have the general solution by making

$$\psi = \phi\{f_1, f_2, \ldots f_i\}$$

$f_1, f_2, \ldots f_i$ being i particular solutions.

Ex. Let it be required to find a symmetrical function of $x_1, x_2, \ldots x_n$, the equations to be satisfied are

$$\psi(x_1, x_2, \ldots x_n) = \psi(x_2, x_3, \ldots x_n, x_1) = \psi(x_3, x_4 \ldots x_n, x_1, x_2) = \&c.$$

or the whole of the conditions may be more concisely denoted by the expression

$$\psi\{\overline{x}_1, \overline{x}_2, \overline{x}_3 \ldots \overline{x}_n\}$$

We may easily find n particular solutions which fulfill these equations: for in the first place it is evident that the sum of

any number of quantities is symmetrical with respect to them, therefore

$$f(x, x, \ldots x) = x + x + \&c. + x = S(x)$$

Again the sum of their products two by two is also symmetrical, therefore

$$f(x, x, \ldots x) = xx + xx + xx + \&c. = S(xx)$$

and similarly the sums of their products three by three, four by four, &c. are symmetrical. We may, therefore, find n different particular solutions, and the general solution will be any arbitrary function of all these particular solutions, or

$$\psi(x, x, \ldots x) = \varphi\left\{S(x), S(xx), \ldots S(xx \ldots x)\right\}$$

Instead of taking for particular solutions the sum of all the quantities, the sum of all the products by two's, the sum of all the products by three's, &c. &c. we might have employed the sum of all the quantities, the sum of their squares, the sum of their cubes, &c. but the solution thus deduced would not be essentially different from the former.

On functional equations of the second and higher orders involving two or more variables.

The notation to be employed in these enquiries has already been sufficiently explained, and the different species of second functions have been noticed. Preserving the same symbols, let it be required to solve the following Problem.

PROBLEM VIII.

Given the equation

$$\phi^{2r+1}(x, y) = x$$

This equation though apparently involving two variables

may in fact be solved by the methods of the first part; for y may be considered as a constant quantity, and if in the solution of $\varphi^{n} x = x$ (Probs. 9, 10, and 14, Part I.) we put arbitrary functions of y instead of the constant quantities which occur, we shall have a solution of the given equation, thus a particular solution of $\varphi^{n} x = x$ is $f(x) = \frac{b-x}{1-cx}$ instead of c put $\chi(y)$ then a solution of the given equation is

$$\psi(x,y) = \frac{b-x}{1-x\chi y}$$

for

$$\psi^{2,1}(x,y) = \psi(\psi x, y), y) \frac{b - \frac{b-x}{1-x\chi y}}{1 - \frac{b-x}{1-x\chi}\chi y} = \frac{x - bx\chi y}{1 - b\chi y} = x$$

We might also instead of b put any other arbitrary function of y, and the result will be the same. The equations

$$\psi^{2,2}(x,y) = \alpha(x,y) \text{ and } \psi^{1,2}(x,y) = \alpha(x,y)$$

may be treated in a similar manner, in the first y must be considered as constant, and x must be so treated in the latter. In general, when functions are taken only relative to one of the variables, the rules delivered in my former Paper are sufficient for their solution, such is the equation

$$F\left\{x, y)\, \psi(x,y), \psi^{2,1}(x,y), \ldots \psi^{n,1}(x,y)\right\} = o$$

It might however occur, that though the order of the function does not vary relative to the other variable, yet that that variable may occur in different forms in each function. An example will render this more evident α, β, &c. being known functions, let

$$F\left\{x, y, \psi(x,y), \psi^{2,1}(x, \alpha y), \psi^{3,1}(x, \beta y), \ldots \psi^{n,1}(x, \nu y)\right\} = o$$

here though the functions do not vary in *order* relative to y,

yet they do vary in a certain sense, because y is differently contained under each functional characteristic; the method of treating these kind of equations will be explained hereafter.

PROBLEM IX.

Given the functional equation

$$\overline{\psi^{2,2}}(x,y) = 0$$

This signifies that the second simultaneous function is equal to zero. It is evident that $x-y$ or $y-x$ will be a particular solution, for if $\psi(x,y) = x-y$ we have

$$\overline{\psi^{2,2}}(x,y) = \psi(\psi(x,y),\psi(x,y)) = (x-y)-(x-y) = 0$$

By observing the process just gone through, it appears that it would equally succeed if for x we put $f(x)$ and for y we put $f(y)$ for if $\psi(x,y) = fx-fy$, we have

$$\overline{\psi^{2,2}}(x,y) = (fx-fy)-(fx-fy) = 0$$

This solution is considerably more general than the former, yet is by no means the complete solution, a more general one may be obtained thus: we found one particular solution to be $\psi(x,y) = x-y$, now if we multiply the right side of this equation by an arbitrary function of x and y the condition will still be fulfilled; for if $\psi(x,y) = (x-y)\varphi(x,y)$ we shall find

$$\overline{\psi^{2,2}}(x,y) = \left\{\overline{x-y}\,\varphi(x,y)-\overline{x-y}\,\varphi(x,y)\right\} \times \varphi\left\{\overline{x-y}\,\varphi(x,y),\overline{x-y}\,\varphi(x,y)\right\} = 0$$

provided $\varphi\left\{\overline{x-y}\,\varphi(x,y),\overline{x-y}\,\varphi(x,y)\right\}$ does not contain in its denominator any factor which vanishes.

PROBLEM X.

Given the equation $\overline{\psi^{2,2}}(x,y)=a$

In this case the second simultaneous function of x and y is constant. The first solution which presents itself is $\psi(x,y)=x-y+A$, then we shall find

$$\overline{\psi^{2,2}}(x,y)=((x-y+A)-(x-y+A)+A=A=a$$

therefore $A=a$ and one particular solution is

$$\psi(x,y)=x-y+a$$

This may be rendered more general, nearly in the same manner as the last Problem; thus let $\psi(x,y)=(x-y)\varphi(x,y)+a$ then

$$\overline{\psi^{2,2}}(x,y)=[(\overline{x-y}\,\varphi(x,y)+a)-(\overline{x-y}\,\varphi(x,y)+a)]\times \varphi\left\{\overline{x-y}\,\varphi(x,y)+a,\overline{x-y}\,\varphi(x,y)+a\right\}+a=a$$

Another particular solution which readily occurs is

$$\psi(x,y)=A\frac{x}{y} \text{ this gives } \overline{\psi^{2,2}}(x,y)=A\frac{A\frac{x}{y}}{A\frac{x}{y}}=A=a$$

therefore $A=a$ and a particular solution is

$$\psi(x,y)=\frac{ax}{y} \text{ or } \psi(x,y)=\frac{ay}{x}$$

this readily points out another general solution, let

$$\psi(x,y)=A\varphi\left(\frac{x}{y}\right) \text{ hence } \overline{\psi^{2,2}}(x,y)=A\varphi\left(\frac{A\varphi\frac{x}{y}}{A\varphi\frac{x}{y}}\right)=A\varphi(1)=a$$

make $A=\frac{a}{\varphi(1)}$ and the general solution is

$$\psi(x,y)=\frac{a}{\varphi(1)}\cdot\left(\frac{x}{y}\right)$$

From the combination of the two preceding solutions we

may obtain another value of ψ which will also satisfy the given equation; it will be

$$\psi(x,y) = \frac{a\,\chi\left\{\overline{x-y}\,\varphi(x,y), \varphi\frac{x}{y}\right\}}{\chi\left\{0, \varphi(1)\right\}}$$

This on trial will be found to agree with the condition, and χ, φ and φ are arbitrary functions

The equation we are considering will also be satisfied by making $\psi(x,y) = a\frac{\varphi(x)}{\varphi(y)}$ or more generally by the constant quantity a multiplied by any fraction whose numerator and denominator become equal when x is put for y: such are the following.

$$a\frac{x+xy+y^2}{y+2x^2},\ a\frac{x(y^2+x^2)}{2y^3},\ a\frac{\sqrt{x^3+8x^2y-5y^3}}{2x\sqrt{x}},\ \&c.$$

PROBLEM XI.

Given the equation

$$\psi^{n,n}(x,y) = ax + by$$

Assume $\psi(x,y) = px + qy$ then we have

$$\psi^{2,2}(x,y) = p(px+qy) + q(px+qy) = (p+q)(px+qy)$$

and $$\psi^{3,3}(x,y) = (p+q)^2(px+qy)$$

and generally

$$\psi^{n,n}(x,y) = (p+q)^{n-1}(px+qy)$$

hence $p.(p+q)^{n-1} = a$ and $q(p+q)^{n-1} = b$, which gives for the values of p and q

$$p = \frac{a}{(a+b)^{\frac{n-1}{n}}} \text{ and } q = \frac{b}{(a+b)^{\frac{n-1}{n}}}$$

This is a very limited solution not containing even an arbitrary constant, it might easily be rendered more general, but

the problem itself would scarcely have been worth noticing had it not been for the very curious results to which it led me.

The relation between $\psi(x, y) = px + qy$ and $\overline{\psi^{n,n}}(x, y) = (p+q)^{n-1}(px+qy)$ is very remarkable, it appears from this, that in the present case in going from the n^{th} to the $n+1^{th}$ simultaneous function, we have only to multiply by the sum of the co-efficients of the original function. On enquiring a little more minutely into the cause of this circumstance, it will be found that it depends on the original function containing x and y of the same dimensions in all its terms, or more generally that the expression of $\psi(x, y)$ is homogeneous. Let us now assume some homogeneous function, and examine its second and higher simultaneous functions, let

$$\psi(x,y) = ax^n + by^p x^{n-p} + cy^q x^{n-q} + \&c.$$

the second simultaneous function is

$$\overline{\psi^{2,2}}(x,y) = a\left\{\psi(x,y)\right\}^n + b\left\{\psi(x,y)\right\}^n + c\left\{\psi(x,y)\right\}^n + \&c.$$

$$\text{or } \overline{\psi^{2,2}}(x,y) = \left\{\psi(x,y)\right\}^n \left\{a+b+c+\&c.\right\} = \psi(1,1)\left\{\psi(x,y)\right\}^n \quad (a)$$

If we now take the simultaneous third functions we have

$$\overline{\psi^{3,3}}(x,y) = \psi(1,1)\left[\psi^{2,2}(x,y)\right]^n = \psi(1,1)\left[\psi(1,1)\left(\psi(x,y)\right)^n\right]^n$$

$$\text{hence} \quad \overline{\psi^{3,3}}(x,y) = \left\{\psi(1,1)\right\}^{1+n}\left\{\psi(x,y)\right\}^{n^2}$$

Repeating the same operation we should have

$$\overline{\psi^{4,4}}(x,y) = \left\{\psi(1,1)\right\}^{1+n+n^2} \times \left\{\psi(x,y)\right\}^{n^3}$$

and generally

$$\overline{\psi^{k,k}}(x,y) = \left\{\psi(x,y)\right\}^{n^{k-1}} \times \left\{\psi(1,1)\right\}^{1+n+\&c.+n^{k-2}} =$$

$$= \left\{\psi(x,y)\right\}^{n^{k-1}} \times \left\{\psi(1,1)\right\}^{\frac{1=n^{k-1}}{1=n}} \quad (b)$$

This elegant property of homogeneous functions will assist us in solving a variety of equations.

PROBLEM XII.

Given the equation

$$\psi^{n,n}(x,y) = a\left\{\psi(x,y)\right\}^{b}$$

Determine n from the equation $b = n^{k-1}$ and also determine $\psi(1,1)$ from the equation $\left\{\psi(1,1)\right\}^{\frac{1-n^{k-1}}{1-n}} = a$

Or the given equation will be satisfied by any homogeneous function of the degree indicated by $b^{\frac{1}{k-1}}$ provided the sum of all its coefficients is equal to the quantity $a^{\frac{1-b}{1-b^{\frac{1}{k-1}}}}$.

Ex. Let $\psi^{3,3}(x,y) = 8(\psi(x,y))^{4}$

here $b = 4, k = 3$, therefore $n^{k-1} = n^{2} = b = 4$ and $n = \pm 2$

also $a = 8$ and $\psi(1,1) = 8^{\frac{1}{3}} = 2$

therefore any of the following quantities will satisfy the equation $2xy$, $x^{2}+y^{2}$, $xy+y^{2}$ $x^{2}-xy+2y^{2}$

The properties of homogeneous functions are so nearly connected with the solution of equations containing simultaneous functions, that it will be convenient to examine into them a little farther, and to adopt some means of denoting them with brevity. In order to signify that a function of several variables has in each of its terms the sum of the indices of any two of them always the same, I shall make use of a line placed beneath those two variables: thus $\psi(\underline{x,y})$ signifies an homogeneous function of x and y; and

as it may be convenient also to denote the sum of the two indices, I shall place it underneath on the outside of the parenthesis; thus then the expression $\psi(\underset{\sim}{x}, \underset{\sim}{y})_q$ denotes any homogeneous function of x and y of the q^{th} degree. A function of three variables x, y, and z, may be homogeneous, with respect to two of them (x and y) in one sense, and also relative to y and z in another; but it does not from thence follow that it will be homogeneous relative to all three, such a function would be denoted thus

$$\psi(\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z})_{p, q}$$

a particular case of this expression is $x^3z^3 + xyz = \psi(\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z})_{2, 2}$.

This notation being premised, we have the following theorems relative to homogeneous functions.

$$\overline{\psi^{n, 1}}(\underset{\sim}{x}, \underset{\sim}{y})_n = \psi(1, 1)\left[\psi(\underset{\sim}{x}, \underset{\sim}{y})_n\right]^n \qquad (1)$$

$$\overline{\psi^{k, k}}(\underset{\sim}{x}, \underset{\sim}{y})_n = \left\{\psi(\underset{\sim}{x}, \underset{\sim}{y})_n\right\}^{n^{k-1}} \times \left\{\psi(1, 1)^{\frac{1-n^{k-1}}{1-n}}\right. \qquad (2)$$

And generally if we have any homogeneous function of the n^{th} degree, and instead of x and y we substitute any other function whatever as $\psi(x, y)$, then we shall have the following equation

$$\psi\left\{\underset{1}{\psi}(x, y), \underset{1}{\psi}(x, y)\right\}_n = \psi(1, 1) \times \left[\psi(x, y)\right]^k \qquad (3)$$

Assume $\psi(x, y) = \varphi\left\{\frac{\alpha(\underset{\sim}{x}, \underset{\sim}{y})_n}{\beta(\underset{\sim}{x}, \underset{\sim}{y})_m}\right\}$

call the latter member, for the sake of brevity, K, and take the second simultaneous function on both sides; in this case $\alpha(\underset{\sim}{x}, \underset{\sim}{y})_n$ will become $\alpha(1, 1)\,K^n$ by eq. (3), and for the same reason $\beta(\underset{\sim}{x}, \underset{\sim}{y})_m$ will become $\beta(1, 1)\,K^m$, and consequently we shall have

$$\overline{\Psi^{2,2}}(x,y)=\Phi\left\{\frac{\alpha(1,1)}{\beta(1,1)}K^{n-m}\right\}=\Phi\left\{\frac{\alpha(1,1)}{\beta(1,1)}\left(\Phi\frac{\alpha(x,y)_n}{\beta(x,y)_m}\right)^{n-m}\right\} \quad (4)$$

let $\alpha(1,1)=\beta(1,1)$ and also let $m=n$, then this equation becomes $\overline{\Psi^{2,2}}(x,y)=\Phi(1)$

this affords another and a more direct solution of Prob. 10. for if $\overline{\Psi^{2,2}}(x,y)=a$, a general solution is

$$\Psi(x,y)=\frac{a}{\Phi(1)}\Phi\left\{\frac{\alpha(x,y)_n}{\beta(x,y)_n}\right\}$$

$\alpha(1,1)$ being equal to $\beta(1,1)$, this latter condition, however, is not absolutely necessary, and if we wish to avoid it, the general solution will be

$$\Psi(x,y)=\frac{a}{\Phi\left(\frac{\alpha(1,1)}{\beta(1,1)}\right)}\Phi\left\{\frac{\alpha(x,y)_n}{\beta(x,y)_n}\right\}$$

the following is another solution of the same question

Assume

$$\Psi(x,y)=\Phi\left\{\begin{array}{c}a+bx+cx^2+\&c.\\ \quad by+cxy\\ \quad\quad\vdots\\ \quad\quad cy^2\\ \quad\quad\vdots\\ \hline {}^1a+{}^1bx+{}^1cx^2+\&c.\\ \quad {}^1by+{}^1cxy\\ \quad\quad\vdots\\ \quad\quad {}^1cy^2\\ \quad\quad\vdots\end{array}\right\}=K$$

taking the second function on both sides we have

$$\overline{\Psi^{2,2}}(x,y)=\Phi\left\{\begin{array}{c}\left.\begin{array}{c}a+b\\ b\\ \vdots\end{array}\right\}K+\left\{\begin{array}{c}c\\ c\\ \vdots\\ c\\ \vdots\end{array}\right\}K^2+\&c.\\ \hline \left.\begin{array}{c}{}^1a+{}^1b\\ {}^1b\\ \vdots\end{array}\right\}K+\left\{\begin{array}{c}{}^1c\\ {}^1c\\ \vdots\\ {}^1c\\ \vdots\end{array}\right\}K^2+\&c.\end{array}\right\}$$

let $a={}^1a \quad b+b_1={}^1b+{}^2b \quad c+c_1+c_2={}^2c+{}^1c+{}^2c$ &c.

then the equation is reduced to

$$\psi^{\overline{2,2}}(x,y) = \Phi(1)$$

If therefore we assume

$$\psi(x,y) = \frac{a}{\varphi(1)}\Phi\left\{\frac{\begin{matrix} a + bx + cx^2 + \&c. \\ by \quad cxy \\ \quad {}^2cy^2 \\ \quad\quad {}^2 \end{matrix}}{\begin{matrix} {}^1a + {}^1bx + {}^1cx^2 + \&c. \\ {}^1by \quad {}^1cxy \\ \quad {}^2 \\ \quad {}^1cy^2 \\ \quad\quad {}^2 \end{matrix}}\right\}$$

the original equation will be satisfied.

I am inclined to think, that this solution is not the most general of which the Problem admits, even though the series were continued back, as it might, to negative powers of x and y. The two solutions which follow are possibly more general, although on this point I am not certain. It would indeed be a very important step, if we could assign the number and nature of the arbitrary functions which enter into the complete solution of functional equations.

Another solution of the equation $\psi^{\overline{2,2}}(x,y) = a$ may be thus deduced, let

$$\psi(x,y) = \varphi\left\{\frac{\alpha(x,y)_m}{{}^1\alpha(x,y)_m}, \frac{\beta(x,y)_n}{{}^1\beta(x,y)_n}, \&c.\right\} = K$$

then taking the second simultaneous function on both sides, it will be perceived by the construction of the second side of the equation, that

$$\psi^{\overline{2,2}}(x,y) = \varphi\left\{\frac{\alpha(1,1)}{{}^1\alpha(1,1)}, \frac{\beta(1,1)}{{}^1\beta(1,1)}, \&c.\right\}$$

call the right side of the equation A, then a very general

solution of our equation is

$$\psi(x, y) = \frac{a}{A}\,\phi\left\{\frac{a(x, y)_m}{a(x, y)_m}, \frac{\beta(x, y)_n}{\beta(x, y)_n}, \text{ \&c.}\right\}$$

where the numbers m, n, p, &c. are not confined to integers. Another solution may be found in the following manner:

let $$\psi(x, y) = \phi\left(\frac{\chi(x, y)}{\chi(x, y)}\right) = K$$

and determine χ so that $\chi(x, y) = \chi(x, y)$ when y is made equal to x, then taking the second simultaneous functions on both sides, we have

$$\overline{\psi^{2,2}}(x, y) = \phi\left(\frac{\chi(K, K)}{\chi(K, K)}\right) = \phi(1)$$

a general solution of the equation in question is therefore

$$\psi(x, y) = \frac{a}{\phi(1)}\,\phi\left(\frac{\chi(x, y)}{\chi(x, y)}\right)$$

this solution depends on that of the equation

$$\chi(x, y) = \chi(x, y) \qquad [y = x]$$

which belongs to a class of equations we shall speak of hereafter.

Let us now return to the consideration of equation (4) it is

$$\overline{\psi^{2,2}}(x, y) = \phi\left\{\frac{a(1, 1)}{\beta(1, 1)}\left[\phi\left(\frac{a(x, y)_n}{\beta(x, y)_m}\right)\right]^{n-m}\right\}$$

for n put $n+1$ and for m put n, then it becomes

$$\overline{\psi^{2,2}}(x, y) = \phi\left\{\frac{a(1, 1)}{\beta(1, 1)}\phi\left(\frac{a(x, y)_{n+1}}{\beta(x, y)_n}\right)\right\}$$

take the third simultaneous function then

$$\overline{\psi^{3,3}}(x, y) = \phi\left\{\frac{a(1, 1)}{\beta(1, 1)}\phi\left[\frac{a(1, 1)}{\beta(1, 1)}\phi\left(\frac{a(x, y)_{n+1}}{\beta(x, y)_n}\right)\right]\right\} \qquad (5)$$

if we suppose $\alpha(1,1) = \beta(1,1)$ these equations become

$$\psi^{\overline{2,2}}(x,y) = \Phi^2\left(\frac{\alpha(\underline{x},\underline{y})_{n+1}}{\beta(\underline{x},\underline{y})_n}\right)$$

$$\psi^{\overline{3,3}}(x,y) = \Phi^3\left(\frac{\alpha(\underline{x},\underline{y})_{n+1}}{\beta(\underline{x},\underline{y})_n}\right)$$

and generally we should find

$$\psi^{\overline{p,p}}(x,y) = \Phi^p\left(\frac{\alpha(\underline{x},\underline{y})_{n+1}}{\beta(\underline{x},\underline{y})_n}\right) \qquad (6)$$

where $\psi(x,y) = \Phi\left(\frac{\alpha(\underline{x},\underline{y})_{n+1}}{\beta(\underline{x},\underline{y})_n}\right)$ and also $\alpha(1,1) = \beta(1,1)$

A more general expression, and one which contains (b) as a particular case may be deduced in the following manner.

Let $$\psi(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\Phi\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right\}^{\frac{1}{n-m}} = K$$

taking the second simultaneous functions on both sides we find

$$\psi^{\overline{2,2}}(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\Phi\frac{\alpha(1,1)K^n}{\beta(1,1)K^m}\right\}^{\frac{1}{n-m}} = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\Phi\frac{\alpha(1,1)}{\beta(1,1)}K^{n-m}\right\}^{\frac{1}{n-m}} =$$

$$= \left\{\frac{\beta(1,1)}{\alpha(1,1)}\Phi\left(\frac{\alpha(1,1)\beta(1,1)}{\beta(1,1)\alpha(1,1)}\Phi\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}} = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\Phi^2\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}}$$

and if we continue to take the succeeding simultaneous functions we shall find generally, that when

$$\psi(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\Phi\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}}$$

$$\psi^{\overline{p,p}}(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\Phi^p\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}} \qquad (7)$$

this expression is much more general than the preceding (b)

with which it coincides when $\alpha(1, 1) = \beta(1, 1)$ and $n = m + 1$ by their assistance we may solve a variety of problems relating to simultaneous functions.

From (b) we have

$$\psi^{2,2}(x, y) = \varphi^2\left(\frac{\alpha(\underline{x}, \underline{y})_{n+1}}{\beta(\underline{x}, \underline{y})_n}\right) = \varphi\,\varphi\left(\frac{\alpha(\underline{x}, \underline{y})_{n+1}}{\beta(\underline{x}, \underline{y})_n}\right)$$

putting in this for $\varphi\left(\frac{\alpha(\underline{x}, \underline{y})_{n+1}}{\beta(\underline{x}, \underline{y})_n}\right)$ its value $\psi(x, y)$ we have

$$\psi^{2,2}(x, y) = \varphi\,\psi(x, y)$$

from this we may deduce the solution of the following Problem.

PROBLEM XIII.

Given the equation

$$\psi^{2,2}(x, y) = \mathrm{F}\,\psi(x, y)$$

make $\varphi = \mathrm{F}$ and take $\alpha(\underline{x}, \underline{y})_{n+1}$ any homogeneous function of the $n+1$*th* degree, and $\beta(\underline{x}, \underline{y})_n$ a similar function of the n^{th}, also let $\alpha(1, 1) = \beta(1, 1)$ then the equation is satisfied by making

$$\psi(x, y) = \mathrm{F}\left\{\frac{\alpha(\underline{x}, \underline{y})_{n+1}}{\beta(\underline{x}, \underline{y})_n}\right\}$$

Ex. Let $\psi^{2,2}(x, y) = \sqrt{\psi(x, y)}$

Suppose $\alpha(\underline{x}, \underline{y})_{n+1} = x^2 + y^2$ and $\beta(x, y)_n = 2x$, then one solution is $\psi(x, y) = \sqrt{\frac{x^2 + y^2}{2x}}$

or let $\alpha(\underline{x}, \underline{y})_{n+1} = (x^2 + y^2)\,\varphi(1)$ and $\beta(\underline{x}, \underline{y})_n = 2y\,\varphi\left(\frac{x}{y}\right)$

then a more general solution is $\psi(x, y) = \sqrt{\frac{x^2 + y^2}{2y\,\varphi\frac{x}{y}}\,\varphi(1)}$

PROBLEM XIV.

Given the equation

$$\psi^{\overline{n,n}}(x,y) = \mathrm{F}\psi(x,y)$$

In equation (6) we have

$$\psi^{\overline{n,n}}(x,y) = \varphi^{n}\left(\frac{\alpha(x,y)_{n+1}}{\beta(x,y)_{n}}\right) = \varphi^{n-1}\varphi\left(\frac{\alpha(x,y)_{n+1}}{\beta(x,y)_{n}}\right)$$

put for $\varphi\left(\frac{\alpha(x,y)_{n+1}}{\beta(x,y)_{n}}\right)$ its value $\psi(x,y)$ then we have

$$\psi^{\overline{n,n}}(x,y) = \varphi^{n-1}\psi(x,y) = \mathrm{F}\psi(x,y)$$

determine φ from the equation $\varphi^{n-1}v = \mathrm{F}v$ by Prob. 13. Part 1. and the general solution of the equation is

$$\psi(x,y) = \varphi\left(\frac{\alpha(x,y)_{n+1}}{\beta(x,y)_{n}}\right)$$

Ex. 1. Let $\psi^{\overline{2,2}}(x,y) = \psi(x,y)$

in this case $\varphi^{n-1}v = \mathrm{F}v$ becomes $\varphi^{2}v = v$ solutions of which are

$$\varphi v = \frac{a}{v}, \quad \varphi v = a - v, \quad \varphi v = (a - v^{m})^{\frac{1}{m}}$$

let $\alpha(x,y)_{n+1} = x^{2} + y^{2}$ and $\beta(x,y)_{n} = x + y$ then solutions of the equation $\psi^{\overline{2,2}}(x,y) = \psi(x,y)$ are

$$\psi(x,y) = \frac{a(x+y)}{x^{2}+y^{2}}, \quad \psi(x,y) = a - \frac{x^{2}+y^{2}}{x+y}$$

$$\text{and } \psi(x,y) = \left\{a - \left(\frac{x^{2}+y^{2}}{x+y}\right)^{m}\right\}^{\frac{1}{m}}$$

Ex. 2. Let $\psi^{\overline{n,n}}(x,y) = \left\{\psi(x,y)\right\}^{m}$

in this example $\varphi^{n-1}v = \mathrm{F}v$ becomes $\varphi^{n-1}v = v^{m}$ and $\varphi v = v^{\frac{1}{m^{n-1}}}$ its particular solutions are therefore

$$\left(\frac{x^{2}+y^{2}}{x+y}\right)^{\frac{1}{m^{n-1}}} \quad \left(\frac{x^{2}+2xy-y^{2}}{y^{2}}\right)^{\frac{1}{m^{n-1}}}$$

other more general ones are

$$\psi(x,y)=\left\{\frac{(x^2+y^2)\,\varphi(1)}{2x\varphi\left(\frac{x}{y}\right)}\right\}^{\frac{1}{n-1}} \text{ and } \psi(x,y)=\left\{\frac{2xy\varphi(1)}{(x^2+y)\varphi\left(\frac{y}{x}\right)}\right\}^{\frac{1}{n-1}}$$

Problem XV.

Required the solution of the equation

$$\psi^{\overline{n,n}}(x,y)=F\psi^{\overline{n-p,n-p}}(x,y)$$

In (6) we have

$$\psi^{\overline{n,n}}(x,y)=\varphi^{n}\left(\frac{\alpha(\underline{x},\underline{y})_{n-1}}{\beta(\underline{x},\underline{y})_{n}}\right)=\varphi^{p}\,\varphi^{n-p}\left(\frac{\alpha(\underline{x},\underline{y})_{n+1}}{\beta(\underline{x},\underline{y})_{n}}\right)$$

but $\psi(x,y)=\varphi\left(\frac{\alpha(\underline{x},\underline{y})_{n+1}}{\beta(\underline{x},\underline{y})_{n}}\right)$ consequently $\varphi^{n-p}\left(\frac{\alpha(\underline{x},\underline{y})_{n+1}}{\beta(\underline{x},\underline{y})_{n}}\right)=\psi^{\overline{n-p,n-p}}(x,y)$ and substituting this value, we shall find

$$\psi^{\overline{n,n}}(x,y)=\varphi^{p}\,\psi^{\overline{n-p,n-p}}(x,y)$$

make $\varphi^{p}v=Fv$ and find the value φ, then the general solution of the equation is $\psi(x,y)=\varphi\left(\frac{\alpha(\underline{x},\underline{y})_{n+1}}{\beta(\underline{x},\underline{y})_{n}}\right)$ if $\alpha(1,1)=\beta(1,1)$.

Ex. 1. Let $\psi^{\overline{n,n}}(x,y)=\left\{\psi^{\overline{2,2}}(x,y)\right\}^{n}$

here $\varphi^{2}v=Fv=v^{n}$ and $\varphi v=v^{n/2}$

therefore $\psi(x,y)=\left\{\frac{\alpha(\underline{x},\underline{y})_{n+1}}{\beta(\underline{x},\underline{y})_{n}}\right\}^{n/2}$ if $\alpha(1,1)=\beta(1,1)$

more particular solutions are

$$\psi(x,y)=\left(\frac{x+y}{2\varphi\frac{y}{x}}\varphi(1)\right)^{n/2} \text{ and } \psi(x,y)=\left(\frac{x^2+2xy+y^2}{3y}\right)^{n/2}$$

PROBLEM XVI.

Given the equation

$$F\left\{\psi(x,y), \psi^{\overline{2,2}}(x,y), \text{ \&c. } \psi^{\overline{p,p}}(x,y)\right\} = 0$$

The same substitution as that employed in the last Problem will reduce this equation of two variables to a similar one which contains only one

putting $\psi(x,y) = \phi\left(\frac{\alpha(x,y)_{n+1}}{\beta(x,y)_n}\right)$ we have $\psi^{\overline{2,2}}(x,y) = \phi^2\left(\frac{\alpha(x,y)_{n+1}}{\beta(x,y)_n}\right)$

and $\psi^{\overline{p,p}}(x,y) = \phi^p\left(\frac{\alpha(x,y)_{n+1}}{\beta(x,y)_n}\right)$ also making $\frac{\alpha(x,y)_{n+1}}{\beta(x,y)_n} = v$

the given equation becomes

$$F\left\{\phi v, \phi^2 v, \ldots \phi^p v\right\} = 0$$

an equation which contains only one variable, and may therefore be solved by the methods described in the first part.

PROBLEM XVII.

Required the solution of the equation

$$F\left\{\psi(x,y), \psi^{\overline{2,2}}(x,y), \ldots \psi^{\overline{p,p}}(x,y)\right\} = 0$$

The following considerations lead to another mode of solution applicable to this Problem. If in the function $\psi(x,y)$ we put y equal to x it becomes $\psi(x,x)$ call this ϕx: then if in the second simultaneous function of $\psi(x,y)$ we put y equal x, the result will be the same as if we had taken the second function of $\psi(x,x)$ or ϕx relative to x, or symbollically expressed, it is

$$\psi^{\overline{2,2}}(x,y) = \psi\left\{\psi(x,x), \psi(x,x)\right\}, = \phi^2 x \quad [y=x]$$

this may be rendered evident by substituting for the right side of the equation its value $\psi(\psi(x,y), \psi(x,y))$.

In the same manner it may be shown, that if we take the p^{th} simultaneous function, and then put x for y, the result will be the same as the p^{th} function of φx, or expressed in symbols it is

$$\psi^{\overline{p,p}}(x,y) = \varphi^p(x) \quad [y=x]$$

Now since this equation is identical when y is equal to x, it will remain so when any other quantity as v is put for x, if the same quantity is also put for y, therefore

$$\psi^{\overline{p,p}}(v,v) = \varphi^p(v)$$

now let $v=\psi(x,y)$ this equation becomes

$$\psi^{\overline{p,p}}(\psi(x,y),\psi(x,y)) = \varphi^p\psi(x,y)$$

but the right side of this equation is nothing more than the $\overline{p+1}th$ simultaneous function of $\psi(x,y)$, consequently

$$\psi^{\overline{p+1,p+1}}(x,y) = \psi^p\psi(x,y)$$

If now in the equation of the Problem we substitute the several values thus formed of the simultaneous functions, we shall have

$$F\left\{\psi(x,y), \varphi\psi(x,y), \varphi^2\psi(x,y), \ldots \varphi^{p-1}\psi(x,y)\right\} = 0$$

and putting x for $\psi(x,y)$ we have

$$F\left\{x, \varphi x, \varphi^2 x, \ldots \varphi^{p-1} x\right\} = 0$$

which is a functional equation of one variable, and may be solved by the methods of the first Part. The form of φ being thus ascertained, we have for determining $\psi(x,y)$ the equation

$$\psi(x,y) = \varphi x \quad [y=x]$$

or expressed in words $\psi(x,y)$ may be any function of x and y which becomes equal to φx when y is equal to x.

PROBLEM XVIII.

Given the equation $\psi^{\overline{p,p}}(x,y) = \mathrm{F}(x,y)$
$\mathrm{F}(x,y)$ being such a function of x and y that it may be reducible to the form $\mathrm{F}\gamma(\underline{x},\underline{y})$.

Assume $\psi(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\varphi\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}}$

then from eq. (7) we have

$$\psi^{\overline{p,p}}(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\varphi^p\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}}$$

make $\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m} = \gamma(\underline{x},\underline{y}) = v$, and since $\gamma(\underline{x},\underline{y})$ is given and α and β are indeterminate, this equation may be easily satisfied in an infinite number of ways; put v for $\gamma(\underline{x},\underline{y})$ our equation becomes

$$\left\{\frac{\beta(1,1)}{\alpha(1,1)}\varphi^p v\right\}^{\frac{1}{n-m}} = \mathrm{F}v$$

which contains only one variable and may be solved by the methods of the former part.

Ex. 1. Let $\psi^{\overline{p,p}}(x,y) = \frac{x+y}{2f(1)}f\left(\frac{x}{y}\right)$

Assume $\alpha(\underline{x},\underline{y}) = (x+y)f\left(\frac{x}{y}\right)$ and $\beta(\underline{x},\underline{y}) = 2f(1)$
then it becomes

$$\varphi^p v = v$$

and calling f any particular solution of this equation we have for the general one

$$\psi(x,y) = \overline{\varphi}^{-1} f\varphi\left(\frac{x+y}{2f(1)}\cdot f\left(\frac{x}{y}\right)\right)$$

Ex. 2. Let $\psi^{\overline{3,3}}(x,y) = \frac{x^3}{y}$

put $\alpha(\underline{x},\underline{y}) = x^3$ and $\beta(\underline{x},\underline{y}) = y$

then the equation to be solved is $\phi^3 v = v$ a particular solution of which is $\phi v = \frac{1+v}{1-3v}$ putting for v its value and using the the general solution we have

$$\psi(x,y) = \overline{\phi}^{-1}\left\{\frac{1+\phi\frac{x^3}{y}}{1-3\phi\frac{x^3}{y}}\right\}$$

as a particular case, take

$$\psi(x,y) = \left\{\frac{y^m+x^{3m}}{y^m-3x^{3m}}\right\}^{\frac{1}{m}}$$

which will be found on trial to satisfy the condition.

Problem XIX.

Given the equation

$$F\left\{\gamma(\underline{x},\underline{y}), \psi(x,y), \psi^{\overline{3,3}}(x,y), \ldots \psi^{\overline{p,p}}(x,y)\right\} = 0$$

This equation is evidently capable of solution by the same means as the last; putting

$$\psi(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\phi\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}}$$

we have as before

$$\psi^{\overline{p,p}}(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\phi^p\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}}$$

and assuming α and β such that $\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m} = \gamma(\underline{x},\underline{y}) = v$ and

making $\frac{\beta(1,1)}{\alpha(1,1)} = a$ our equation becomes

$$F\left\{v, (a\varphi v), \overset{\frac{1}{n-m}}{(a\varphi^{a}v)}, \overset{\frac{1}{n-m}}{\ldots} (a\varphi^{r})^{\frac{1}{n-m}}\right\} = 0$$

which may be solved by Prob. XIX. Part I.

PROBLEM XX.

Given the equation

$$\psi^{2,1}(x,y) = \psi^{1,2}(x,y)$$

This equation, containing no simultaneous function, is different from any we have yet solved, and requires the application of a peculiar artifice.

In my former Paper, in order to reduce the equation $\psi^{2}x = ax$ to one of the first order, I made use of the substitution $\overrightarrow{\varphi} f \varphi x$ for ψx: an analogous one must be employed on the present occasion; let us suppose

$$\psi(x,y) = \overrightarrow{\varphi} f(\varphi x, \varphi y)$$

the effect of this will be very similar to that of the one just alluded to, and its great utility will be evident by considering its result in the various orders of the same function, thus

$$\psi^{2,1}(x,y) = \overrightarrow{\varphi} f(\varphi \overrightarrow{\varphi} f(\varphi x, \varphi y), \varphi y) = \overrightarrow{\varphi} f(f(\varphi x, \varphi y), \varphi y)$$
$$= \overrightarrow{\varphi} f^{2,1}(\varphi x, \varphi y)$$

$$\psi^{1,2}(x,y) = \overrightarrow{\varphi} f(\varphi x, \varphi \overrightarrow{\varphi} f(\varphi x, \varphi y)) = \overrightarrow{\varphi} f(\varphi x, f(\varphi x, \varphi y))$$
$$= \overrightarrow{\varphi} f^{1,2}(\varphi x, \varphi y)$$

$$\psi^{2,\overline{2}}(x,y) = \overrightarrow{\varphi} f(\varphi \overrightarrow{\varphi} f(\varphi x, \varphi y), \varphi \overrightarrow{\varphi} f(\varphi x, \varphi y)) = \overrightarrow{\varphi} f(f(\varphi x, \varphi y),$$
$$f(\varphi x, \varphi y)) = \overrightarrow{\varphi} f^{\overline{2,2}}(\varphi x, \varphi y)$$

and continuing the same substitutions we shall find

$$\psi^{3,1}(x,y) = \overrightarrow{\varphi} f^{3,1}(\varphi x, \varphi y) \text{ and } \psi^{1,3}(x,y) = \overrightarrow{\varphi} f^{1,3}(\varphi x, \varphi y)$$

and generally

$$\psi^{n,m}(x,y) = \overline{\varphi}^{1} f^{n,m}(\varphi x, \varphi y)$$

If there are more variables than two, the proper substitution to be made is

$$\psi(x, x, \ldots x) = \overline{\varphi}^{1} f^{1,1,\&c.}(\varphi x, \varphi x, \ldots \varphi x)$$

and there would result generally

$$\psi^{n,m,p,\&c.}(x, x, \ldots x) = \overline{\varphi}^{1} f^{n,m,p,\&c.}(\varphi x, \varphi x, \ldots \varphi x)$$

By such substitutions all simple functional equations of every order and of any number of variables, may be reduced to those of the first order: but the difficulty is not then overcome, the resulting equations are by no means easy to solve, and in a variety of cases it appears, that they are contradictory or impossible.

Let us apply this substitution to the equation of this Problem, then since $\psi(x,y) = \overline{\varphi}^{1} f(\varphi x, \varphi y)$ we have

$$\overline{\varphi}^{1} f^{2,1}(\varphi x, \varphi y) = \overline{\varphi}^{1} f^{1,2}(\varphi x, \varphi y)$$

Performing the operation denoted by $\overline{\varphi}^{1}$ on both sides it becomes

$$f^{2,1}(\varphi x, \varphi y) = f^{1,2}(\varphi x, \varphi y)$$

Put $\overline{\varphi}^{1} x$ for x and $\overline{\varphi}^{1} y$ for y then it becomes

$$f^{2,1}(x,y) = f^{1,2}(x,y)$$

which is nothing more than the original equation; from it, however we learn, that if we can find one particular solution, we can always deduce from it the general one, which supposing f a particular case, will be

$$\psi(x,y) = \overline{\varphi}^{1} f(\varphi x, \varphi y)$$

After repeated endeavours I have been unable to find any particular case which will satisfy the equation

$$\psi^{2,1}(x,y) = \psi^{1,2}(x,y)$$

I have also made some attempts at discovering particular solutions of the two following equations, and have met with no success.

$\psi^{\overline{2,2}}(x,y) = \psi^{1,2}(x,y)$ and $\psi^{\overline{2,2}}(x,y) = \psi^{2,1}(x,y)$

Should, however, any particular case be found, their general solutions flow immediately from the method just explained.

With regard to the equation of the Problem $\psi^{2,1}(x,y) = \psi^{1,2}(x,y)$, I have little expectation of finding any particular case, as I think the following reasoning, though perhaps not quite so satisfactory as might be wished, will show the impossibility of it. First, let us suppose, that $\psi(x,y)$ is a symmetrical function of x and y, let it be $\chi(\bar{x},\bar{y})$ then our equation becomes

$$\chi\{\overline{\chi(\bar{x},\bar{y})},\bar{y}\} = \chi\{\bar{x},\overline{\chi(\bar{x},\bar{y})}\} = \chi\{\overline{\chi(\bar{x},\bar{y})},\bar{x}\}$$

Comparing the first of these expressions with the third, we may observe that in the first, wherever $\chi(\bar{x},\bar{y})$ occurs, the same quantity $\chi(\bar{x},\bar{y})$ also occurs in the third, consequently in this respect, the first and third are identical: but wherever y occurs in the first, x occurs similarly in the third, therefore in this respect they cannot be identical, unless y is equal to x. From this it appears, that the equation in question cannot be solved by any symmetrical function. Again, the given equation $\psi^{2,1}(x,y) = \psi^{1,2}(x,y)$ contains x and y in the same manner, and no reason can be assigned why in the solution x should be contained differently from y: this may, perhaps, be made more clear, thus. Let $f(x,y)$ be the quantity to which each side of the given equation is equal, then

$$\psi^{2,1}(x,y) = f(x,y) = \psi^{1,2}(x,y)$$

Now since $\psi(\psi(x,y),y) = f(x,y)$ and also $\psi(x,\psi(x,y)) =$

$f(x,y)$; and since taking the second functions is a direct operation, it is evident that the original function $\psi(x,y)$ will produce the same result, whether we take the second function relative to x or relative to y; therefore it must be similarly composed of x and y; that is to say, it must be symmetrical relative to x and y: but we have before shown that no symmetrical function can satisfy the equation, consequently the equation is contradictory.

This train of reasoning I offer with considerable hesitation, well aware of the extreme difficulty of reasoning correctly on a subject so very general, and which, from its novelty, the mind has not been sufficiently habituated to consider, so as to rely with confidence on any lengthened process of reasoning. I thought it, however, right to mention this proof, that those who may seek for particular cases, might first enquire whether the equation be possible.

Problem XXI.

Given the equation

$$x\,\psi^{1,2}(x,y) = y\,\psi^{2,1}(x,y)$$

Substituting $\overline{\varphi}^{1} f(\varphi x, \varphi y)$ for $\psi(x,y)$ in this equation we have

$$x\,\overline{\varphi}^{1} f^{1,2}(\varphi x, \varphi y) = y\,\overline{\varphi}^{1} f^{2,1}(\varphi x, \varphi y)$$

putting $\overline{\varphi}^{1} x$ for x and $\overline{\varphi}^{1} y$ for y it becomes

$$\overline{\varphi}^{1} x \,.\, \overline{\varphi}^{1} f^{1,2}(x,y) = \overline{\varphi}^{1} y \,.\, \overline{\varphi}^{1} f^{2,1}(x,y)$$

This equation will be satisfied if we could find such a form for f, that the two following equations might be fulfilled.

$$f^{1,2}(x,y) = y \text{ and } f^{2,1}(x,y) = x$$

for in that case it would become

$$\overline{\varphi}^{1} x \,.\, \overline{\varphi}^{1} y = \overline{\varphi}^{1} y \,.\, \overline{\varphi}^{1} x$$

which is identical.

Our enquiries must therefore be directed to this point, and it will be found that $f(x,y) = a - x - y$ has the required properties, and is a particular solution of the given equation: hence the general solution is

$$\psi(x,y) = \overline{\varphi}^{1}(a - \varphi x - \varphi y)$$

There are many other particular cases which fulfil the same condition, such as

$$f(x,y) = \frac{a}{xy} \text{ and } f(x,y) = \frac{1-x-y}{1-bxy}$$

these give the general solutions

$$\psi(x,y) = \overline{\varphi}^{1}\frac{a}{\varphi x\,\varphi y} \text{ and } \psi(x,y) = \overline{\varphi}^{1}\left(\frac{1-\varphi x-\varphi y}{1-b\varphi x\,\varphi y}\right)$$

PROBLEM XXII.

Given the equation

$$\psi^{2,1}(x,y) \,.\, \psi^{1,2}(x,y) = xy$$

Using the same substitution employed in the last Problem, this equation becomes

$$\overline{\varphi}^{1} f^{2,1}(\varphi x, \varphi y) \,.\, \overline{\varphi}^{1} f^{2,1}(\varphi x, \varphi y) = xy$$

and putting $\overline{\varphi}^{1} x$ for x, and $\overline{\varphi}^{1} y$ for y we have

$$\overline{\varphi}^{1} f^{2,1}(x,y) \,.\, \overline{\varphi}^{1} f^{1,2}(x,y) = \overline{\varphi}^{1} x \,.\, \overline{\varphi}^{1} y$$

which becomes identical, if $f^{2,1}(x,y) = x$ and $f^{1,2}(x,y) = y$ consequently all the solutions of the last Problem also solve this.

PROBLEM XXIII.

Given the equation

$$\mathrm{F}\left\{x, \psi^{1,2}(x,y)\right\} = \mathrm{F}\left\{\psi^{2,1}(x,y), y\right\}$$

this equation may be solved by the same artifice as the two last, assuming $\psi(x,y) = \overline{\varphi}^{1} f(\varphi x, \varphi y)$ we have

$$\mathrm{F}\left\{x, \overline{\varphi}^{1} f^{1,2}(\varphi x, \varphi y)\right\} = \mathrm{F}\left\{\overline{\varphi}^{1} f^{2,1}(\varphi x, \varphi y), y\right\}$$

and putting $\overline{\varphi}^{1} x$ for x, and $\overline{\varphi}^{1} y$ for y, it becomes

$$\mathrm{F}\left\{\overline{\varphi}^{1} x, \overline{\varphi}^{1} f^{1,2}(x,y)\right\} = \mathrm{F}\left\{\overline{\varphi}^{1} f^{2,1}(x,y), \overline{\varphi}^{1} y\right\}$$

this is identical if we assume f so that the two conditions $f^{1,2}(x,y) = y$ and $f^{2,1}(x,y) = x$ may be fulfilled.

The same method is applicable to the equation

$$\left\{\psi^{2,1}(x,y) - x\right\} \mathrm{F}(x, y, \psi(x,y) \text{ \&c.}) = \left\{\psi^{1,2}(x,y) - y\right\} \mathrm{F}(x, y, \psi(x,y), \text{ \&c.})$$

for the two factors which multiply F and F vanish on account of the value of f.

PROBLEM XXIV.

Given the equation

$$x\psi^{2,2}(x,y) = a\psi^{2,1}(x,y)$$

this equation, by means of the substitution already so frequently employed, becomes

$$x\overline{\varphi}^{1} f^{2,2}(\varphi x, \varphi y) = a\overline{\varphi}^{1} f^{2,1}(\varphi x, \varphi y)$$

and putting $\overline{\varphi}^{1} x$ for x, and $\overline{\varphi}^{1} y$ for y we have

$$\overline{\varphi}^{1} x . \overline{\varphi}^{1} f^{2,2}(x,y) = a\overline{\varphi}^{1} f^{2,1}(x,y)$$

An artifice somewhat similar to the one already employed, will afford the solution of this equation: if we can find such a value of $f(x,y)$ that $f^{2,2}(x,y) = c$ and also $f^{2,1}(x,y) = x$ the equation will become identical by making $c = \varphi a$. Such a value of f is $f(x,y) = c\frac{y}{x}$ for

$$f^{2,2}(x,y) = c\frac{c\frac{y}{x}}{c\frac{y}{x}} = c \text{ and } f^{2,1}(x,y) = c\frac{\frac{y}{x}}{c\frac{y}{x}} = x$$

hence the general solution of the given equation is

$$\psi(x,y) = \overline{\varphi}^{1}\left(\frac{\varphi a \varphi y}{\varphi x}\right)$$

PROBLEM XXV.

Given the equation

$$F\left\{\psi^{2,2}(x,y), x\right\} = F\left\{a, \psi^{2,1}(x,y)\right\}$$

making the substitution $\overline{\varphi}^{1} f(\varphi x, \varphi y)$ for $\psi(x,y)$ and in the result putting $\overline{\varphi}^{1} x$ for x, and $\overline{\varphi}^{1} y$ for y, we shall find

$$F\left\{\overline{\varphi}^{1} f^{2,2}(x,y), \overline{\varphi}^{1} x\right\} = F\left\{a, \overline{\varphi}^{1} f^{2,1}(x,y)\right\}$$

which will become identical if we select such a value for $f(x,y)$ that $\overline{\varphi}^{1} f^{2,2}(x,y) = a$, and also $f^{2,1}(x,y) = x$ such a value is $f(x,y) = \varphi a + y - x$; hence the general solution of the Problem is

$$\psi(x,y) = \overline{\varphi}^{1}(\varphi a + \varphi y - \varphi x)$$

the more general equation

$$\left\{\psi^{2,1}(x,y) - x\right\} F\left\{x, y, \psi(x,y) \text{ \&c.}\right\} = \psi^{2,2}(x,y) \cdot F\left\{x, y, \psi(x,y), \text{ \&c.}\right\}$$

may be solved nearly in the same manner, its solution will be

$$\psi(x,y) = \overline{\varphi}^{1}(\varphi y - \varphi x)$$

PROBLEM XXVI.

Given the equation

$$F\left\{x, y, \psi(x,y), \psi^{1,2}(x,y), \psi^{2,1}(x,y), \text{ \&c.}\right\} = 0$$

Assume $\psi(x,y) = \overline{\varphi}^{1} f(\varphi x, \varphi y)$ then we have $\psi^{2,1}(x,y) = \overline{\varphi}^{1} f^{2,1}(\varphi x, \varphi y)$, and $\psi^{1,2}(x,y) = \overline{\varphi}^{1} f^{1,2}(x,y)$ and generally $\psi^{n,m}(x,y) = \overline{\varphi}^{1} f^{n,m}(\varphi x, \varphi y)$

Substituting these values the equation becomes

$$F\left\{x, y, \overline{\varphi}^{1} f(\varphi x, \varphi y), \overline{\varphi}^{1} f^{1,2}(\varphi x, \varphi y), \overline{\varphi}^{1} f^{2,1}(\varphi x, \varphi y), \text{ \&c.}\right\} = 0$$

and putting $\overline{\varphi}^{1}x$ for x and $\overline{\varphi}^{1}y$ for y we have

$$F\left\{\varphi\overline{\varphi}^{1}x, \varphi\overline{\varphi}^{1}y, \varphi\overline{\varphi}^{1}f(x,y), \varphi\overline{\varphi}^{1}f^{1,2}(x,y), \varphi\overline{\varphi}^{1}f^{2,1}(x,y), \&c.\right\}=0$$

Some particular value of f must now be assumed, and the equation treated as one of the first order relative to $\varphi\overline{\varphi}^{1}$.

The form to be assigned to f is of some consequence; it ought to be a particular solution of the original equation: for if we assign to it any other form, this adds a limitation to the original equation which may or may not agree with it, a particular solution should therefore always be employed. This remark is applicable to several Problems in my former Paper, and with this restriction, their solutions will remain correct.

Problem XXVII.

To transform the equation

$$F\left\{x, y, \psi(x,y), \psi^{2,1}(\alpha x, \beta y), \psi^{1,2}(\underset{1}{\alpha x}, \underset{1}{\beta y}), \&c.\right\}=0$$

into the form of the equation of the preceding Problem.

Assume $\psi(x,y)=\overline{\Phi}^{1}f(\Phi x, \Phi y)$, then the equation becomes

$$F\left\{x, y, \overline{\varphi}^{1}f(\varphi x,\varphi y), \overline{\varphi}^{1}f^{2,1}(\varphi\alpha x, \varphi\beta y), \overline{\varphi}^{1}f^{1,2}(\varphi\underset{1}{\alpha x}, \varphi\underset{1}{\beta y}), \&c.\right\}=0$$

find for φ by Prob. VII. Part I. such a value that it shall not change when any of the following quantities are substituted for x.

αx	βx
$\underset{1}{\alpha x}$	$\underset{1}{\beta x}$
$\underset{2}{\alpha x}$	$\underset{2}{\beta x}$
&c.	&c.

let this value be A, then the equation becomes

$$F\left\{x, y, \overline{A}^{1}f(Ax, Ay,) \overline{A}^{1}f^{2,1}(Ax, Ay), \&c.\right\}=0$$

or putting $\overline{A}^{x}x$ for x, and $\overline{A}^{y}y$ for y we have

$$F\left\{\overline{A}^{x}x, \overline{A}^{y}y, \overline{A}^{y}f(x,y), \overline{A}^{y}f^{2,1}(x,y), \overline{A}^{y}f^{1,2}(x,y), \&c.\right\} = 0$$

an equation of the required form.

One important use of this transformation is the solution of equations of the form

$$F\left\{x, y, \psi(x,y), \psi^{2,1}(\alpha x, y), \psi^{1,2}\beta x, y), \&c.\right\} = 0$$

in which the functions are taken only relative to y, and yet x is not altogether constant. It may be transformed into

$$F\left\{x, y, \psi(x,y), \psi^{1,2}(x,y), \psi^{1,3}(x,y), \&c.\right\} = 0$$

which may be treated as an equation of one variable, x being constant. This is the species of equation alluded to at page (198)

After considering the various equations amongst the higher orders of functions, another question presents itself, which may be thus stated. What must be the form of a function of (n) variables, such that taking the functions relative to any or to all of them any number of times, and combining these quantities in any manner, the result shall (when all these variables are made equal to x) be equal to a given function of x? This question might thus be expressed when there are only two variables

$$F\left\{x, y, \psi(x,y), \psi^{1,2}(x,y), \psi^{2,1}(x,y), \&c.\right\} = f(a) \quad [y = x]$$

this condition obviously enlarges the signification of the function ψ, and the solutions ought to be more general. We shall accordingly find that some equations, of which without this condition we cannot find even a particular solution, are capable, when it is added, of very extensive ones. When there are more than two variables, the condition may be, that making

them equal by pairs, the result shall be given: a particular case would be the equation

$$F\{x,y,z,v,w,r,\psi(x,y,z,v,w,r,\&c.\}=F\{x,v,r\}\begin{bmatrix}y=x\\z=v\\w=r\end{bmatrix}$$

Instead of making y equal to x, and z to v and so on, y might become a given function of x, and z a given function of v, &c. thus:

$$F\{x,y,z,v,w,r,\psi(x,y,z,v,w,r,),\&c.\}=F\{x,v,r,\}\begin{bmatrix}y=\alpha x\\z=\beta v\\w=\gamma r\end{bmatrix}$$

a few examples will sufficiently explain the method to be pursued in treating these equations.

Problem XXVIII.

Given the equation

$$\psi^{2,1}(x,y)=\psi^{1,2}(x,y)\qquad[y=x]$$

This Problem, of which without the condition of y being made equal to x we could not find even a particular case, readily admits of solution in its present state. Since $\psi^{2,1}(x,y)$ is only equal to $\psi^{1,2}(x,y)$, when $y=x$ we may put for the given equation

$$\psi^{2,1}(x,y)=\varphi\{(x,y),\psi^{1,2}(x,y)\}\qquad(1)$$

provided that when y is equal to x, the latter side of the equation shall become $\psi^{1,2}(x,y)$, and this fully satisfies the condition of the Problem. If, therefore, we can find such a value of φ, the equation (1) may be treated as a common functional equation of two variables, and may be solved by the rules already given.

Nor is it at all difficult to find such a value of φ; if we make $\psi^{1,2}(x,y)=z$, φ must be such a function that

$$\varphi(x,y,z)=z\qquad[y=x]$$

It is evident that particular values of φ are

$$x - y + z \quad \text{and} \quad \frac{x}{y} z$$

many others might be mentioned, but it is desirable to determine φ more generally

Since $\varphi(x, y, z) = z \quad [y = x]$

it is evident that $\varphi(x, x, z) = z$

and since this is independent on any particular value of x we have

$$\varphi(v, v, z) = z$$

that is to say, that whatever quantity is substituted for x, if the same quantity is also substituted for y, the result will be equal to z. Now let $v = \varphi(x, y, z)$, it becomes

$$\varphi\{\varphi(x, y, z), \varphi(x, y, z), z\} = z$$

but this expression is nothing more than the second simultaneous function relative to x and y, and may be therefore more concisely expressed thus

$$\varphi^{\overline{2,2},1}(x, y, z) = z$$

in which equation, since it does not vary relative to z, that quantity may be considered as a constant; and the equation

$$\varphi^{\overline{2,2}}(x, y) = z = \text{constant}$$

being solved, we have only to substitute instead of the various constant quantities arbitrary functions of z: thus then the solution of the equation

$$\varphi(x, y, z) = x \quad [y = x]$$

is reduced to that of

$$\psi^{\overline{2,2}}(x, y) = \text{constant}$$

and we have only to refer to Problem (10) for its general solution.

Let us apply this to the solution of the equation of this Problem $\psi^{2,1}(x, y) = \psi^{1,2}(x, y) \quad [y = x]$

take as a particular case of the equation $\varphi^{2,2,1}(x,y,z)=z$ $\varphi(x,y,z)=\frac{x}{y}z$, then the equation of the Problem becomes

$\psi^{2,1}(x,y)=\frac{x}{y}\psi^{1,2}(x,y)$ or $\quad y\psi^{2,1}(x,y)=x\psi^{1,2}(x,y)$

this is the equation solved in Prob. XXI. therefore all its solutions are also solutions of this equation. This however is, comparatively speaking, but a very limited answer: every different solution of the equation $\varphi^{2,2,1}(x,y,z)=z$ furnishes a new solution of our Problem, containing one or more arbitrary functions; each of these may very justly be called a general solution; but to investigate the number and nature of the arbitrary constants which enter into the complete solution, is an enquiry of considerable difficulty.

PROBLEM XXIX.

Given the equation

$$\psi^{\left\{\begin{smallmatrix}2,2\\1,2\end{smallmatrix}\right\}^n}(x,y)=\mathrm{F}(x) \qquad [y=ax]$$

This signifies, that after taking the second function relative to x, and then the second relative to y; the result is considered merely as a function of x, and its n^{th} function taken relative to that variable: lastly, the quantity to which this becomes equal, after performing these operations, is given. The manner of treating these equations is very simple; put

$\psi^{2,2}(x,x)=\chi(x)$, then our equation becomes

$$\psi^{\left\{\begin{smallmatrix}2,2\\1,2\end{smallmatrix}\right\}^n}(x,y)=\chi^n(x)=\mathrm{F}(x) \qquad [y=ax]$$

determine χ from the equation $\chi^n x=\mathrm{F}(x)$ by Prob. XIII. Part I. and let its solution be $\overset{1}{\mathrm{F}}(x)$, then we have

$$\psi^{2,2}(x,y)=\overset{1}{\mathrm{F}}(x) \qquad [y=ax]$$

This equation may be solved by nearly the same method as that employed in the last Problem.

If the function occurs in different shapes or of various orders, this method is inapplicable, as in the following Problem.

PROBLEM XXX.

Given the equation

$$F\left\{\psi^{2,1)3}(x,y), \psi^{1,2)3}(x,y), x, y\right\} = 0 \quad [y = x]$$

The difficulties in this case appear to be much encreased from this circumstance, that the second function of $\psi^{2,1}(x,x)$ relative to x is quite different from the second function of $\psi^{1,2}(x,x)$ relative to the same quantity. The method of solution which I shall explain is equally applicable to all of this species, and consists in reducing them to a class which has been already solved.

It may be observed, that whether we take the second function of $\psi^{2,1}(x,x)$ relative to x, or whether we take the simultaneous function of $\psi^{2,1}(x,y)$ considered as a simple function, and in the result put x for y, the two expressions will be the same; the first gives

$$\psi^{2,1}(\psi^{2,1}(x,x), \psi^{2,1}(x,x))$$

and the second is

$$\psi^{2,1}(\psi^{2,1}(x,y), \psi^{2,1}(x,y))$$

which when y becomes equal to x is identical with the former; but

$$\psi^{2,1}(\psi^{2,1}(x,y), \psi^{2,1}(x,y)) = \psi^{2,1}\left\{\psi(\psi(x,y),y), \psi(\psi(x,y),y)\right\} = \overset{2,1,1,1,1}{\underset{1\ 2\ 1\ 2\ 1}{\psi}}(x,y)$$

the lower line of indices denoting the quantities relative to which the operations are performed. In a similar manner it may be shown, that

$$\overset{1,2)3}{\psi}(x,y) = \overset{1,2,2,2,2}{\underset{1,2,1,2,1}{\psi}}(x,y)$$

Substituting these values in the original equation, we have

$$\mathrm{F}\left\{\overset{1,1,\ 1,1,\ 2}{\psi}(x,y), \overset{1,2,\ 1,2,\ 1}{\psi}(x,y), x, y\right\} = 0 \quad [y = x]$$

This is an equation similar to that of Problem XXVIII., and may be solved by the same means.

New methods of solving functional equations of the first order, and also differential functional equations.

The new methods which I now propose to explain are only applicable to equations of the form

$$\mathrm{F}\left\{x, \psi x, \psi \alpha x, \psi \alpha^2 x, \ldots \psi \alpha^n x\right\} = 0$$

where α must be such a function that $\alpha^{n+1}x = x$. By the method of Prob. VII. Part I. all functional equations of the first order may be reduced to this form; and although in many cases this reduction is very difficult, or even in the present state of analysis out of our power, yet it is theoretically possible, and we shall therefore consider all equations as so reduced. There is this remarkable difference between the former methods and the present one:

Those which I have already given always led to the general solution, and perhaps, in some cases, to the complete one; these, on the contrary, which I shall now propose, always conduct us directly to a particular solution, which does not contain even an arbitrary constant. It has, however, several advantages; it is the most direct method with which we are yet acquainted; and if by any means we could introduce into these solutions an arbitrary constant, it would afford us general ones: this is a step which is wanting to connect it with the former methods. In the case of differential functional

equations, this step is supplied by the integrations which are necessary, and we thus arrive at their general solutions.

PROBLEM XXXI.

Given the equation

$$\mathrm{F}\left\{x, \psi x, \psi \alpha x\right\} = 0$$

and also $\alpha^{n} x = x$

Find ψx in terms of x and $\psi \alpha x$; let it be

$$\psi x = \underset{1}{\mathrm{F}}\left\{x, \psi \alpha x\right\}$$

put αx for x; then it becomes

$$\psi \alpha x = \underset{1}{\mathrm{F}}\left\{\alpha x, \psi \alpha^{2} x\right\} = \mathrm{F}\left\{\alpha x, \psi x\right\}$$

put this value of $\psi \alpha x$ in the former equation, and we have

$$\psi x = \underset{1}{\mathrm{F}}\left\{x, \underset{1}{\mathrm{F}}\left(\alpha x, \psi x\right)\right\}$$

from which equation ψx may be found in terms of x.

If $\alpha^{3} x = x$ instead of $\alpha^{2} x = x$, we should find

$$\psi x = \underset{1}{\mathrm{F}}\left\{x, \underset{1}{\mathrm{F}}\left\{\alpha x, \underset{1}{\mathrm{F}}\left\{\alpha^{2} x, \psi x\right\}\right\}\right\}$$

Ex. 1. Take the equation $(\psi x)^{p} \cdot \psi(a - x) = x^{n}$ where $\alpha x = a - x$

then $$\psi x = \left(\frac{x^{n}}{\psi(a-x)}\right)^{\frac{1}{p}}$$

and $$\psi(a - x) = \left\{\frac{(a-x)^{p}}{\psi x}\right\}^{\frac{1}{p}}$$

this substituted in the former gives

$$\psi x = \left\{\frac{x^{n}}{\left(\frac{(a-x)^{p}}{\psi x}\right)^{\frac{1}{p}}}\right\}^{\frac{1}{p}}$$

from which we find

$$\psi x = \left\{\frac{x}{(a-x)^{\frac{1}{p}}}\right\}^{\frac{np}{p^{2}-1}}$$

which will be found on trial to satisfy the equation.

Ex. 2. Given the equation $x^n \psi x - a\psi \frac{1}{x} = bx^p$ here $\alpha x = \frac{1}{x}$ and $\alpha^2 x = x$, and we have

$$\psi x = \frac{bx^p - a\psi\frac{1}{x}}{x^n}$$

and putting $\frac{1}{x}$ for x it becomes

$$\psi \frac{1}{x} = \frac{b\overrightarrow{x^p} - a\psi x}{\overrightarrow{x^n}}$$

Substituting this in the former equation we find for the value of ψx

$$\psi x = \frac{b}{1-a^2}\left\{ x^{p-n} + a\overrightarrow{x^p} \right\}$$

Ex. 3. Given the equation $x^n \psi x - x^m \psi\left(\frac{a-x}{1-cx}\right) = x^p$ by employing the same method its solution will be

$$\psi(x) = \frac{x^m\left(\frac{a-x}{1-cx}\right)^p + x^p\left(\frac{a-x}{1-cx}\right)^m}{\left(x\frac{a-x}{1-cx}\right)^n - \left(x\frac{a-x}{1-cx}\right)^m}$$

Problem XXXII.

Given the equation

$$F\left\{x, \psi x, \psi \alpha x, \ldots \psi \alpha^n x\right\} = 0$$

and also $\alpha^{n+1} = x$

putting successively x, αx, $\alpha^2 x$, &c. $\alpha^n x$ for x, we have the following equations:

$$F\left\{x, \psi x, \psi \alpha x, \ldots \psi \alpha^n x\right\} = 0 \qquad (1)$$

$$F\left\{\alpha x, \psi \alpha x, \psi \alpha^2 x, \ldots \psi \alpha^n x, \psi x\right\} = 0 \qquad (2)$$

&c. &c.

$$F\left\{\alpha^n x, \psi \alpha^n x, \psi x, \psi \alpha x, \ldots \psi \alpha^{n-1} x\right\} = 0 \quad (n+1)$$

From these $n+1$ equations we may eliminate the n quantities $\psi\alpha x$, $\psi\alpha^2 x$, and $\psi\alpha^n x$, and there will remain an equation of the form

$$F\left\{x, \alpha x, \alpha^2 x, \ldots \alpha^n x, \psi x\right\} = 0$$

from which ψx may readily be found.

Ex. 1. Let $\psi x + \psi\alpha x = fx$ and $\alpha^2 x = x$

then $\psi\alpha x + \psi\alpha^2 x = f\alpha x$

also $\psi\alpha^2 x + \psi x = f\alpha^2 x$

hence we find $\psi x = \frac{1}{2}(fx - f\alpha x + f\alpha^2 x)$

Ex. 2. take the equation

$$\psi x + fx\,\psi\alpha x = \underset{1}{f}x$$

where f and $\underset{1}{f}$ are perfectly arbitrary and $\alpha^2 x = x$; then making use of process above described we find

$$\psi x = \frac{\underset{1}{f}x - fx\,f\alpha x}{1 - f\alpha x\, f\alpha^2 x}$$

if $\alpha^3 x = x$ we should have

$$\psi x = \frac{\underset{1}{f}x - fx\,\underset{1}{f}\alpha x + fx\,f\alpha x\,\underset{1}{f}\alpha^2 x}{1 - fx\,f\alpha x\,f\alpha^2 x}$$

and generally when $\alpha^n x = x$ we shall have

$$\psi x = \frac{\underset{1}{f}x - fx\,\underset{1}{f}\alpha x + fx\,f\alpha x\,\underset{1}{f}\alpha^2 x - \&c. \pm fx\,f\alpha x \ldots f\alpha^{n-2} x\,\underset{1}{f}\alpha^{n-1} x}{1 - fx\,f\alpha x\,f\alpha^2 x \ldots f\alpha^{n-1} x}$$

Ex. 3. Take the equation

$$\psi x\,\psi\alpha x + fx\,\underset{1}{\psi}x = fx$$

if $\alpha^2 x = x$, ψx must be deduced from the equation

$$\psi x = \cfrac{\underset{1}{f}x}{\cfrac{\underset{1}{f}\alpha x}{fx + \cfrac{\underset{1}{f}\alpha x}{f\alpha x + \psi x}}}$$

and generally when $a^n x = x$ the form of ψx is determined by the equation.

$$\psi x = \cfrac{fx}{\cfrac{f_a x}{fx + \cfrac{f_a{}^2 x}{f_a x + \cfrac{f_a{}^3 x}{f_a{}^2 x + \&c. \cfrac{+f_a{}^{n-1} x}{f_a{}^{n-1} x + \psi x}}}}}$$

It may be observed, that this method of discovering particular solutions by elimination, will not apply when the given equation contains only the different forms of the function without the variable quantity itself: thus it is not applicable to the equation

$$F\left\{\psi x, \psi a x, \ldots \psi a^n x\right\} = o$$

the reason of this is obvious; for if we eliminate from this equation (by means of the n equations which arise by changing the order in which the functions are placed), all the functions but ψx, we shall have a result containing nothing but ψx and constant quantities, and therefore, ψx is equal to a constant quantity: it is true such a value of ψx will satisfy the equation, but it scarcely deserves the name of a solution.

Another exception is, when the equation

$$F\left\{x, \psi x, \psi a x, \&c. \psi a^n x\right\} = o$$

is homogeneous relative to the different forms of the unknown function; for in this case when we attempt to eliminate them, they all disappear together, leaving an equation of condition; thus given

$$\psi x = (a - x)\,\psi a x \text{ and } a^n x = x$$

we have $\psi a x = (a - ax)\,\psi a^2 x = (a - ax)\,\psi x$

and $\psi x = (a - x)(a - ax)\,\psi x$ or $1 = (a - ax)(a - a^2 x)$

which equation is not necessarily true.

Another exception is, when the given equation can be made to assume the form

$$\mathrm{F}\left\{\overline{\psi x}, \overline{\psi a\, x} \,.\,.\, \overline{\psi a^n x}\right\} = fx$$

In this case the equation cannot be fulfilled unless fx is a symmetrical function of x, ax, &c. $a^n x$, because the first side is such a symmetrical function: this reason, however, should be received with caution, for if the operation denoted by ψ be an inverse one, it may admit of several values, and it seems *possible*, that in such a case the condition relative to the form of f need not be fulfilled. In my former paper I explained the means of finding solutions of the equation $\psi^n x = x$. I then contented myself with explaining the theory without mentioning particular cases; as these latter may be required in our present enquiry, I shall subjoin the following particular solutions of $\psi^2 x = x$

$$\psi x = a - x \qquad \psi x = \log(a - \varepsilon^x) \qquad \psi x = (a^n - x^n)^{\frac{1}{n}} \qquad \psi x = \sqrt{1 - x^2}$$

$$\psi x = \frac{x-2}{x-1} \qquad \psi x = x - \log(\varepsilon^x - 1) \qquad \psi x = \frac{x}{(ax^n - 1)^{\frac{1}{n}}} \qquad \psi x = \frac{a}{x}$$

$$\psi x = \frac{1-x}{1+x} \qquad \psi x = \tan^{-1}(a - \tan x) \qquad \psi x = \frac{x}{x-1}$$

$$\psi x = \frac{a - bx}{b + cx} \qquad \psi x = \tan^{-1}\left(\frac{\sin(a-x)}{\cos a \cos x}\right) \qquad \psi x = \frac{x}{\sqrt{x^2 - 1}}$$

Particular cases of $\psi^3 x = x$

$$\psi x = \frac{a^2}{a - x} \qquad \psi x = \frac{\sqrt{ax^2 - a^2}}{x} \qquad \psi x = \frac{a + bx}{c - \frac{b^2 + bc + c^2}{a}x}$$

$$\psi x = \frac{a^2}{ax - c^2 x} \qquad \psi x = \frac{(ax^n - a^n)^{\frac{1}{n}}}{x} \qquad \psi x = \log(a\varepsilon^x - a^2) - x$$

$$\psi x = \frac{ax - a^2}{x} \qquad \psi x = \left(\frac{a^2}{a - x^2}\right)^{\frac{1}{n}} \qquad \psi x = \log(\varepsilon^x - \varepsilon^c) - x + c$$

$$\psi x = \frac{1+x}{1-3x} \qquad \psi x = \frac{1}{1-x} \qquad \psi x = -\log(1 - \varepsilon^x)$$

Particular cases of $\psi^{s} x = x$

$$\psi x = \frac{1}{2}\,\frac{1}{1-x} \qquad \psi x = \frac{1+x}{1-x} \qquad \psi x = \left(\frac{2}{2-x^{n}}\right)^{\frac{1}{n}}$$

$$\psi x = \frac{2}{2-x} \qquad \psi x = \frac{2a^{2}}{2ac-c^{2}x} \qquad \psi x = \frac{(2x^{n}-3)^{\frac{1}{n}}}{x}$$

$$\psi x = 2\,\frac{x-1}{x} \qquad \psi x = \frac{a+bx}{c-\frac{b^{2}+c^{2}}{2a}x} \qquad \psi x = \log 2 - x + \log(\epsilon^{x}-1)$$

Particular cases of $\psi^{s} x = x$

$$\psi x = \frac{1}{3(1-x)} \qquad \psi x = \frac{3x-1}{3x} \qquad \psi x = \frac{a+bx}{c-\frac{b^{2}-bc+c^{2}}{3a}x}$$

$$\psi x = \frac{3}{3-x} \qquad \psi x = \frac{3a^{2}}{3ac-c^{2}x} \qquad \psi x = \frac{1}{x}\left(x^{n}-\frac{1}{3}\right)^{\frac{1}{n}}$$

$$\psi x = 3\,\frac{x-1}{x} \qquad \psi x = \frac{3+3x}{3-x} \qquad \psi x = \log 3 - x + \log(\epsilon^{x}-1)$$

Problem XXXIII.

Given the equation

$$\psi\alpha x = \frac{d\psi x}{dx}$$

α being such a function that $\alpha^{s} x = x$

For x put αx, then

$$\psi\alpha^{s} x = \psi x = \frac{d\psi\alpha x}{d\alpha x}$$

by differentiating this we have

$$\frac{d\psi x}{dx} = \frac{d.}{dx}\cdot\frac{d\psi\alpha x}{d\alpha x} \qquad (a)$$

but the left side of this equation is by the Problem equal to $\psi\alpha x$; therefore

$$\psi\alpha x = \frac{d.}{dx}\cdot\frac{d\psi\alpha x}{d\alpha x}$$

but we also have

$$\frac{d\psi\alpha x}{dx} = \frac{d\psi\alpha x}{d\alpha x}\cdot\frac{d\alpha x}{dx}$$

consequently

$$\frac{d\psi\alpha x}{d\alpha x} = \frac{d\psi\alpha x}{dx}\cdot\left(\frac{d\alpha x}{dx}\right)^{-1}$$

this being substituted in (a) gives

$$\psi_a x = \frac{d}{dx} \cdot \left\{ \frac{d \psi_a x}{dx} \left(\frac{d \psi_a x}{dx} \right)^{-1} \right\}$$

put $\psi_a x = z$ then

$$z = \frac{d.}{dx} \cdot \left(\frac{dx}{dz} \left(\frac{d_a x}{dx} \right)^{-1} \right)$$

which is a differential equation, from whose solution z or $\psi_a x$ may be found.

Ex. 1. Given the equation

$$\psi (a - x) = \frac{d \psi x}{dx}$$

in this case $\frac{d_a x}{dx} = -1$, and the differential equation is

$$z\, d x^2 + d^2 z = 0$$

its integral is $z = \psi (a - x) = b \cos x + c \sin x$.

The two constant quantities which have entered by integration must be determined so as to satisfy the original equation. This condition gives

$$c = \frac{-b \cos a}{1 - \sin a}$$

the quantity b still remaining arbitrary; the solution of the equation $\psi (a - x) = \frac{d \psi x}{dx}$ is therefore

$$\psi x = b \cos (a - x) - \frac{b \cos a}{1 - \sin a} \sin (a - x)$$

Ex. 2. Take the equation

$$\psi \frac{1}{x} = \frac{d \psi x}{dx}$$

in this case $_a x = \frac{1}{x}$ and $\frac{d_a x}{dx} = \frac{-1}{x^2}$

and the differential equation becomes

$$2\, d x^2 + 2 x\, d x\, d z + x^2 d^2 z = 0$$

whose solution is

$$z = \psi \frac{1}{x} = -\frac{b}{2p+1} x^{-p-1} + \frac{b}{2} x^p \qquad p = \frac{-1 \pm \sqrt{-3}}{2}$$

and $\psi x = -\frac{b}{2p+1} x^{p+1} + \frac{b}{2} x^{-p}$

in order to determine the constants b and $b_{,}$, substitute this expression in the given equation and it will be found that

$$-\frac{b}{2p+1} = -bp \text{ therefore}$$

$$\psi x = -b_{,}px^{p+1} + b_{,}x^{-p}$$

in which there still remains one arbitrary constant.

It is observable that both these solutions contain one constant. Let us suppose this to be changed into an arbitrary function of x, and let us determine what conditions it must be subject to, that it may satisfy the Problem: taking the second example we have

$$\psi x = (x^{-p} - px^{p+1})\,\varphi x$$

and the equation becomes

$$(x - px^{-p-1})\varphi\tfrac{1}{x} = (x^{-p} - px^{p+1})\tfrac{d\varphi x}{dx} + \tfrac{d(x^{-p} - px^{p+1})}{dx}\varphi x$$

from this equation φx must be determined (the method of doing which will appear in a subsequent Problem). If this solution contains an arbitrary constant, the same process may be again repeated. We may thus continue deducing one solution from another as long as we can solve the differential equations to which they give rise, but still these will only be particular solutions.

PROBLEM XXXIV.

Given the equation

$$\psi \alpha x = \tfrac{d^n \psi x}{dx^n}$$

and $\alpha^p x = x$, put for x successively αx, $\alpha^2 x$, . . $\alpha^{p-1} x$ then we have

$$\psi a\, x = \tfrac{d^n \psi x}{dx^n}$$

$$\psi\alpha^{n}x = \frac{d^{n}\psi\alpha x}{(d\alpha x)^{n}}$$

&c. &c.

$$\psi\alpha^{p-1}x = \frac{d^{n}\psi\alpha^{p-2}x}{(d\alpha^{p-2}x)^{n}}$$

$$\psi\alpha^{p}x = \frac{d^{n}\psi\alpha^{p-1}x}{(d\alpha^{p-1}x)^{n}}$$

but $\psi\alpha^{p}x = \psi x$ and combining all these equations we have

$$\psi\alpha^{p}x = \psi x = \frac{d.^{n}}{(d\alpha^{p-1}x)^{n}} \cdot \frac{d.^{n}}{(d\alpha^{p-2}x)^{n}} \cdots \frac{d^{n}\psi x}{(dx)^{n}}$$

which is a differential equation of the pn^{th} order and putting $\psi x = z$ we have

$$z = \frac{d^{np}z}{(d\alpha^{p-1}x.\, d\alpha^{p-2}x. \ldots . d\alpha x.\, dx)^{n}}$$

this being integrated gives the value of z or ψx.

PROBLEM XXXV.

Given the equation

$$F\left\{x, \psi x, \psi\alpha x, \frac{d\psi x}{dx}\right\} = 0 \quad \text{also } \alpha^{p}x = x$$

Find the value of $\psi\alpha x$ from this equation and substitute in it αx, $\alpha^{2}x$, &c. $\alpha^{p-1}x$ for x, then we have

$$\psi\alpha x = F\left\{x, \psi x, \frac{d\psi x}{dx}\right\}$$

$$\psi\alpha^{2}x = F\left\{\alpha x, \psi\alpha x, \frac{d\psi\alpha x}{d\alpha x}\right\}$$

&c. &c.

$$\psi\alpha^{p}x = F\left\{\alpha^{p-1}x, \psi\alpha^{p-1}x, \frac{d\alpha^{n-1}x}{d\alpha^{p-1}x}\right\}$$

In each of these equations for $\frac{d\psi\alpha x}{d\alpha x}$, $\frac{d\psi\alpha^{2}x}{d\alpha^{2}x}$, &c. $\frac{d\psi\alpha^{p-1}x}{d\alpha^{p-1}x}$

put their values $\frac{d\psi\alpha x}{dx}\left(\frac{d\alpha x}{dx}\right)^{-1}$, $\frac{d\psi\alpha^{2}x}{dx}\left(\frac{d\alpha^{2}x}{dx}\right)^{-1}$, &c. $\frac{d\psi\alpha^{p-1}x}{dx}\left(\frac{d\alpha^{p-1}x}{dx}\right)^{-1}$

and also differentiate the results. Then we shall have the two following sets of equations

$$\psi x = \mathrm{F}_{,}\left\{x, \psi x, \frac{d\psi x}{dx}\right\} \qquad (1)$$

$$\psi\alpha x = \mathrm{F}_{,}\left\{\alpha x, \psi\alpha x, \frac{d\psi\alpha x}{dx}\left(\frac{d\alpha x}{dx}\right)^{-1}\right\} \qquad (2)$$

&c. &c.

$$\psi\alpha^{p-1}x = \mathrm{F}_{,}\left\{\alpha^{p-2}x, \psi\alpha^{p-2}x, \frac{d\psi\alpha^{p-2}x}{dx}\left(\frac{d\alpha^{p-2}x}{dx}\right)^{-1}\right\} \qquad (p-1)$$

$$\psi\alpha^{p}x = \mathrm{F}_{,}\left\{\alpha^{p-1}x, \psi\alpha^{p-1}x, \frac{d\psi\alpha^{p-1}x}{dx}\left(\frac{d\alpha^{p-1}x}{dx}\right)^{-1}\right\} \qquad (p)$$

and also

$$\frac{d\psi\alpha x}{dx} = \frac{d.}{dx}\mathrm{F}_{,}\left\{x, \psi x, \frac{d\psi x}{dx}\right\} \qquad (\alpha, 1)$$

$$\frac{d\psi\alpha^{2}x}{dx} = \frac{d.}{dx}\mathrm{F}_{,}\left\{\alpha x, \psi\alpha x, \frac{d\psi\alpha x}{dx}\left(\frac{d\alpha x}{dx}\right)^{-1}\right\} \qquad (\alpha, 2)$$

&c. &c.

$$\frac{d\psi\alpha^{p-1}x}{dx} = \frac{d.}{dx}\mathrm{F}_{,}\left\{\alpha^{p-1}\psi\alpha^{p-2}x, \frac{d\psi\alpha^{p-2}x}{dx}\left(\frac{d\alpha^{p-2}x}{dx}\right)^{-1}\right\} \qquad (\alpha, p-1)$$

$$\frac{d\psi\alpha^{p}x}{dx} = \frac{d.}{dx}\mathrm{F}_{,}\left\{\alpha^{p-1}x, \psi\alpha^{p-1}x, \frac{d\psi\alpha^{p-1}x}{dx}\left(\frac{d\alpha^{p-1}x}{dx}\right)^{-1}\right\} \qquad (\alpha, p)$$

Since $\alpha^{p}x = x$ equation (p) becomes

$$\psi x = \mathrm{F}_{,}\left\{\alpha^{p-1}x, \psi\alpha^{p-1}x, \frac{d\psi\alpha^{p-1}x}{dx}\left(\frac{d\alpha^{p-1}x}{dx}\right)^{-1}\right\}$$

from this by means of equations $(p-1)$ and $(\alpha, p-1)$ we may eliminate $\psi\alpha^{p-1}x$, and $\frac{d\psi\alpha^{p-1}x}{dx}$ the resulting equation will contain only $x, \psi x, \psi\alpha x$, &c. $\psi\alpha^{p-2}x$ and their differentials. From this by means of $(p-2)$ and $(\alpha, p-2)$ we may eliminate $\psi\alpha^{p-2}x$, and its differential, leaving an equation containing only $x, \psi x, \psi\alpha x$, &c. $\psi\alpha^{p-3}x$ and their differentials. In the same manner $\psi\alpha^{p-3}x$ may be eliminated, and the process may be continued

until the last equation will only contain x, ψx and their differentials; this equation must be integrated, and it will determine the value of ψx in terms of x.

The same method may be employed for the solution of the much more general equation

$$\mathrm{F}\left\{ x, \psi x, \psi\alpha x, \ldots \psi\alpha^{p-1}x, \frac{d^n \psi x}{dx^n}, \frac{d^n \psi\alpha^2 x}{dx^n}, \text{\&c.} \right\} = 0$$

provided also, that $\alpha^p x = x$.

By substituting successively for x the quantities αx, $\alpha^2 x$, &c. $\alpha^{p-1}x$, we shall have p equations containing the functions ψx, $\psi\alpha x$, and $\psi\alpha^{p-1}x$ and their differentials.

Let each of these be differentiated as often as may be required, and we shall have two sets of equations by means of which all the quantities except x and ψx, and their differentials may be eliminated, the result is a common differential equation whose integral will afford the value of ψx in terms of x. If after satisfying the conditions of the Problem, there remain any arbitrary constants, we may suppose them functions of x, and new equations will thence arise by which they may be determined.

It might occur (when there are several arbitrary quantities) that, by assigning particular values to some of them, the others might remain in a certain degree arbitrary, should this be the case, we should obtain general solutions.

PROBLEM XXXVI.

Given the equation

$$\psi^2 x = \frac{d\psi x}{dx}$$

Assume $\psi x = \overrightarrow{\varphi} f \varphi x$, then the equation becomes

$$\overrightarrow{\varphi} f^2 \varphi x = \frac{d\overrightarrow{\varphi} f \varphi x}{dx}$$

putting $\overset{-1}{\phi}$ for x we have

$$\overset{-1}{\phi} f' x = \frac{d\overset{-1}{\phi} fx}{d\overset{-1}{\phi} x} = \frac{d\overset{-1}{\phi} fx}{dx} \left(\frac{d\overset{-1}{\phi} x}{dx}\right)^{-1}$$

or $$\overset{-1}{\phi} f' x . \frac{d\overset{-1}{\phi} x}{dx} = \frac{d\overset{-1}{\phi} fx}{dx}$$

Some particular solution of the original equation must now be assumed as the value of f, and the resulting differential functional equation must be solved. The only particular case of the equation $\psi' x = \frac{d\psi x}{dx}$ with which I am at present acquainted, is

$$\psi x = \left(\frac{1 \pm \sqrt{-3}}{2}\right)^{\frac{1 \mp \sqrt{-3}}{2}} \times x^{\frac{1 \pm \sqrt{-3}}{2}}$$

Other more complicated equations containing the various orders of functions, and their differentials may be reduced to those of the first order by the same means, but great difficulties still remain; it is by no means easy to discover particular solutions of the original equations, and even when these are found, the functional equations of the first order which remain to be solved, are of considerable difficulty. I shall therefore refrain from giving any more examples, and proceed to show how functional equations involving definite integrals may be reduced to those we have already treated. Such equations might occur in a variety of curious and interesting enquiries, few of which have yet been noticed. D'Alembert, in one of the volumes of his Opuscules, has examined a question which may be referred to this class; it is the following. Suppose a sphere composed of particles of matter, what must be the law of attraction amongst these particles, so that the force of the whole sphere acting on a particle at a distance, may follow the same law? the question might be varied by supposing the law to be given, and the form of the solid to be required;

but the general solution of such questions is by no means easy.

PROBLEM XXXVII.

Required the nature of the function ψ such that

$$\int dx\, \psi^{x} = \psi a$$

the integral being taken between the limits $x=o$ and $x=a$.

Assume $\varphi(x, v)$ such that

$$\varphi(x, v) - \varphi(o, v) = v$$

the form of φ may be ascertained from this equation by means already described. Then if we make

$$\int dx\, \psi^{x} = \varphi(x, \psi a) \qquad (1)$$

it is evident that between the two limits $x = o$ and $x = a$, the integral will be reduced to ψa, and we have therefore a differential functional equation whose mode of solution has already been pointed out. Other more complicated equations may be solved in the same way; these I shall omit. I shall, however, make some observations on this method of solution, with a view to point out some questions of considerable importance.

In equation (1) the function indicated by φ is so assumed that we may have

$$\varphi(a, \psi a) - \varphi(o, \psi a) = \psi a$$

from which, perhaps, it might be imagined, that $\varphi(x, \psi a)$ must contain only x, ψa and constant quantities, but the condition would still be fulfilled if it contained $\psi^{2}a$, $\psi^{3}a$, or $\psi^{n}a$, which though not actually variable cannot strictly be regarded as constant. To fix our ideas, let us consider the example in this Problem; one value of $\varphi(x, \psi a)$ is evidently $\varphi(x, \psi a) = \frac{x^{2}}{a^{2}}\psi a$, we have therefore

$$\int dx\, \psi^{x} = \frac{x}{a^{2}}\psi a$$

and by differentiating

$$\psi^{s}x = \frac{nx^{n-1}}{a^{n}}\psi a$$

from which ψx may be found.

This is a solution derived from a certain form attributed to φ, but we might also give to φ the form

$$\varphi(x, \psi a) = \frac{x^{n}}{a^{n}}\psi a + x^{p}(x-a)^{q} f(a, \psi a, \psi^{2}a, \ldots \psi^{n}a)$$

and, in that case, the equation to be solved would be

$$\psi^{s}x = \frac{nx^{n-1}}{a^{n}}\psi a + \frac{d.}{dx}\left(x(x-a)^{q}\right) f(a, \psi a, \psi^{2}a, \ldots \psi^{n}a)\Big)$$

this contains only the second function of the unknown quantity and must be solved as a second functional equation, considering $a, \psi a$, &c. $\psi^{n}a$ as constant quantities; let its solution be

$$\psi x = \mathrm{F}\left\{x, a, \psi a, \ldots \psi^{n}a\right\} \qquad (a)$$

then we must put $x=a$ and determine ψa from the equation

$$\psi a = \mathrm{F}\left\{a, a, \psi a, \ldots \psi^{n}a\right\}$$

the value of ψa thus deduced, will furnish the values of $\psi^{2}a$, &c. $\psi^{n}a$, and these being substituted in a, will give the value of ψx; this solution is evidently of a different nature from the former, and forms another species.

Again, the following form of φ will also agree with the conditions

$$\varphi(x, \psi a) = \frac{x^{n}}{a^{n}}\psi a + x^{p}(x-a)^{q} f\left\{a, \psi a, \psi^{2}a, \ldots \psi^{n}a, x, \psi x, \psi^{2}x, \ldots \psi^{t}x\right\}$$

which being substituted in the Problem ψx must be found from a functional equation of the k^{th} order; x must then be put equal to a, and the new functional equation of the n^{th} order relative to a must be solved; this is a third species of solution different from either of the former. Respecting these three species of solutions, a very important question

may be proposed. What degree of generality does each possess, and how many and what sort of arbitrary functions does each solution involve? To discuss this question, and to point out the nature of other solutions yet more general, which may be found for these and other similar Problems, would far exceed the limits of a mere outline of the calculus. I shall conclude my remarks on this Problem by stating the plan to be pursued in one particular case, which may serve as a model for all similar operations. Take as the form of $\varphi(x, \psi a)$

$$\varphi(x, \psi a) = x(x-a)\frac{\psi x\, \psi^2 x}{\psi^2 a\, \psi^2 a} - \frac{x-x}{a}\frac{\psi x}{\psi o}\psi a$$

then we have

$$\psi^2 x = \frac{d.}{dx}\left\{x(x-a)\frac{\psi x\, \psi^2 x}{\psi^2 a\, \psi^2 a} - \frac{a-x}{a}\frac{\psi x}{\psi o}\psi a\right\}$$

this is a differential functional equation which must be solved on the hypothesis of a, ψa, $\psi^2 a$, ψo and $\psi^2 o$, being constant quantities. Let its solution be

$$\psi x = \mathrm{F}\left\{x, a, \psi a, \psi^2 a, \psi o, \psi^2 o\right\} \qquad (1)$$

we must now put $x=a$ and treat the resulting equation as one of the second order, considering ψo and $\psi^2 o$ as constants. Let its solution be

$$\psi a = \mathrm{F}\left\{a, \psi o, \psi^2 o\right\} \qquad (2)$$

Now substitute o for a retaining o as a letter instead of making it actually zero, there will result a new functional equation of the second order, whose solution is

$$\psi o = \mathrm{F}\left\{o\right\}$$

and lastly, substituting this value of ψo, and also that of $\psi^2 o$ which may be deduced from it, in (2) we have the value of ψa, from this $\psi^2 a$ may be found, and these being substituted in (1) give the value of ψx.

Problem XXXVIII.

Given the equation

$$\psi(x,y) = \frac{d\psi(x,\alpha y)}{dx}$$

where $\alpha^{\prime} y = y$.

For y put αy and the equation becomes

$$\psi(x,\alpha y) = \frac{d\psi(x,\alpha^{\prime} y)}{dx} = \frac{d\psi(x,y)}{dx}$$

differentiate this relative to x, then we have

$$\frac{d\psi(x,\alpha y)}{dx} = \frac{d^{\prime}\psi(x,y)}{dx^{\prime}}$$

this substituted in the original equation, gives

$$\psi(x,y) = \frac{d^{\prime}\psi(x,y)}{dx^{\prime}}$$

which is a partial differential equation, whose solution is

$$\psi(x,y) = \varepsilon^{x}\varphi y + \varepsilon^{-x}\varphi y$$

φy and φy being two arbitrary functions of y, so constituted as to fulfil the original equation. These may thus be determined, since

$$\psi(x,y) = \varepsilon^{x}\varphi y + \varepsilon^{-x}\varphi y$$

we have $$\frac{d\psi(x,\alpha y)}{dx} = \varepsilon^{x}\varphi\alpha y - \varepsilon^{-x}\varphi\alpha y$$

and, since these two quantities must be equal, we have the following equations

$$\varphi y = \varphi\alpha y \text{ and } \varphi y = -\varphi\alpha y$$

the former of these is easily satisfied by putting for φy any symmetrical function of y and αy; and a particular solution of the latter is

$$\varphi y = (-y + \alpha y)\, c$$

and since this solution contains an arbitrary constant, it may

be changed by Prob. VIII. Part I. into any arbitrary function which does not vary when y becomes αy; its general solution is therefore

$$\varphi y = (-y + \alpha y)\, \varphi\, (\overline{y}, \overline{\alpha y})$$

and consequently the general solution of the equation of this Problem is

$$\psi\,(x, y) = \varepsilon^{x}\, \varphi\, (\overline{y}, \overline{\alpha y}) + \overline{\varepsilon}^{x}\, (\alpha y - y)\, \varphi_{1}\, (\overline{y}, \overline{\alpha y})$$

Ex. 1. Take the equation

$$\psi\,(x, y) = \frac{d\psi\,(x,\, a-y)}{dx}$$

in this case $\alpha y = a - y$, and the general solution is

$$\psi\,(x, y) = \varepsilon^{x}\, \varphi\, (\overline{y}, \overline{a-y}) + \overline{\varepsilon}^{x}\, (a - 2y)\, \varphi_{1}\, (\overline{y}, \overline{a-y})$$

Ex. 2. Let the given equation be

$$\psi\,(x, y) = \frac{d\psi\,(x,\, \frac{1}{y})}{dx}$$

here $\alpha y = \frac{1}{y}$ and the general solution is

$$\psi\,(x, y) = \varepsilon^{x}\, \varphi\, (\overline{y}, \overline{\tfrac{1}{y}}) + \overline{\varepsilon}^{x}\, \tfrac{1-y^{2}}{y}\, \varphi_{1}\, \left(\overline{y}, \overline{\tfrac{1}{y}}\right)$$

PROBLEM XXXIX.

Given the equation

$$\psi\,(x, y) = \frac{d\psi\,(x,\, \alpha y)}{dx}$$

supposing $\alpha^{p} y = y$.

By substituting successively αy, $\alpha^{2} y$, &c. $\alpha^{p-1} y$ for y, we have the following equations

$$\psi\,(x,\ y) = \frac{d\psi\,(x,\, \alpha y)}{dx}$$

$$\psi\,(x, \alpha y) = \frac{d\psi\,(x,\, \alpha^{2} y)}{dx}$$

&c. &c.

$$\psi\,(x, \alpha^{p-1} y) = \frac{d\psi\,(x,\, \alpha^{p} y)}{dx}$$

From the first of these we may eliminate $\frac{d\psi(x,\alpha y)}{dx}$ by means of the differential of the second, and from the result $\frac{d\psi(x,\alpha^2 y)}{dx}$ may be eliminated by means of the differential of the third. And by continuing this process, observing that $\psi(x,\alpha^p y) = \psi(x,y)$ we shall find

$$\psi(x,y) = \frac{d^p\psi(x,y)}{dx^p}$$

this partial differential equation must be solved, and the arbitrary functions which enter into its integral, must be made to satisfy the conditions of the Problem.

Ex. Let $p=4$, then $\psi(x,y) = \frac{d\psi(x,\alpha y)}{dx}$ and the solution of the resulting partial differential equation will be

$$\psi(x,y) = \varepsilon^x \underset{11}{\varphi} y - \varepsilon^{-x} \underset{12}{\varphi} y + \sin x \cdot \underset{13}{\varphi} y + \cos x \cdot \underset{14}{\varphi} y$$

hence

$$\frac{d\psi(x,\alpha y)}{dx} = \varepsilon^x \underset{11}{\varphi}\alpha y + \varepsilon^{-x} \underset{12}{\varphi}\alpha y + \cos x \cdot \underset{13}{\varphi}\alpha y - \sin x \cdot \underset{14}{\varphi}\alpha y$$

the first condition to be satisfied is

$$\underset{11}{\varphi} y = \underset{12}{\varphi}\alpha y$$

which is readily fulfilled by making $\underset{11}{\varphi} y = \underset{1}{\varphi}(y, \alpha y, \alpha^2 y, \alpha^3 y)$, the next condition is

$$\underset{13}{\varphi} y = -\underset{13}{\varphi}\alpha y$$

This must be solved by Prob. VIII. Part I., and we shall have

$$\underset{13}{\varphi} y = (-y + \alpha y - \alpha^2 y + \alpha^3 y)\, \underset{2}{\varphi}(y, \alpha y, \alpha^2 y, \alpha^3 y)$$

the third and fourth conditions are

$$\underset{13}{\varphi} y = -\underset{14}{\varphi}\alpha y \text{ and } \underset{14}{\varphi} y = \underset{13}{\varphi}\alpha y$$

In the second of these put αy for y, and it becomes $\underset{14}{\varphi}\alpha y = \underset{13}{\varphi}\alpha^2 y$,

this substituted in the former, gives

$$\underset{13}{\varphi} y = -\underset{13}{\varphi} \alpha^3 y$$

whose general solution being found by the method in the first part gives

$$\underset{13}{\varphi} y = (-y + \alpha^3 y)\ \underset{13}{\varphi}\left(\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}\right)$$

and consequently

$$\underset{14}{\varphi} y = (-\alpha y + \alpha^3 y)\ \underset{2}{\varphi}\left(\overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}, \overline{y}\right)$$

these values being respectively substituted, we have for the general solution of the Problem in this example,

$$\psi(x,y) = \varepsilon^x \varphi\left(\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}\right) + \overline{\varepsilon}^x (-y + \alpha y - \alpha^2 y + \alpha^3 y)\underset{5}{\varphi}\left(\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}\right) +$$

$$+ (-y + \alpha^2 y)\underset{8}{\varphi}\left(\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}\right) \sin x + (-\alpha y + \alpha^3 y)\underset{3}{\varphi}\left(\overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}, \overline{y}\right) \cos x$$

If the original equation had been

$$\psi(x,y) = \frac{d^n \psi(x, \alpha y)}{dx^n}$$

the partial differential equation to be solved would have been

$$\psi(x,y) = \frac{d^{np} \psi(x,y)}{dx^{np}}.$$

This form is rather remarkable, the equation can always be integrated when np is a whole number; let us suppose n to be a fraction and p a whole number, some multiple of the denominator of n.

Ex. Let $n = \frac{1}{2}$, $p = 2$, then $np = 1$, and $\alpha^2 y = y$, and the equation to be solved is

$$\psi(x,y) = \frac{d^{\frac{1}{2}} \psi(x, \alpha y)}{dx^{\frac{1}{2}}}$$

whose solution is $\psi(x,y) = \varepsilon^x \underset{1}{\varphi} y$, or by assigning a proper form to $\underset{1}{\varphi} y$ it becomes

$$\psi(x,y) = \varepsilon^x \varphi\left(\overline{y}, \overline{\alpha y}\right)$$

Not only may the index of differentiation become fractional,

but the index of the order of a function may be a fraction* or even a variable quantity, and such equations as the following might occur

$$\frac{d^{\frac{1}{2}}\downarrow^{\frac{1}{2}}z}{\sqrt{dz}} = \frac{d^{\frac{1}{2}}\downarrow^{\frac{1}{2}}z}{dz^{\frac{1}{2}}}$$

To notice the extreme difficulty of the enquiries to which such equations would lead, might seem superfluous, though it may not be deemed equally so to support my own opinion of their utility by the authority of one well acquainted with these subjects. LACROIX, in the third volume of his Traité du Calcul, Diff. et Int. speaking of fractional indices of differentiation, observes, "L'Analyse offre une foule d'expressions de ce genre, qui tiennent presque toutes aux théories les plus importantes et les plus délicates, et les réflexions que j'ai exposées dans le No. 965, me portent à croire que leur considération peut contribuer beaucoup aux progrès de la science du calcul."

PROBLEM XL.

Given the equation

$$\frac{d\downarrow(x,\beta y)}{dx} = \frac{d\downarrow(\alpha x,y)}{dy}$$

also $\alpha^{n}x = x$ and $\beta^{n}y = y$.

Put αx for x, and βy for y, then the equation becomes

$$\frac{d\downarrow(\alpha x,y)}{d\alpha x} = \frac{d\downarrow(x,\beta y)}{d\beta y}$$

$$\text{hence } \frac{d\downarrow(\alpha x,y)}{dx}\left(\frac{d\alpha x}{dx}\right)^{-1} = \frac{d\downarrow(x,\beta y)}{dy}\left(\frac{d\beta y}{dy}\right)^{-1}$$

differentiate this equation relative to y, and the original one relative to x: then the two results are

$$\frac{d^{2}\downarrow(\alpha x,y)}{dxdy} = \frac{d\alpha x}{dx}\frac{d.}{dy}\left\{\frac{d\downarrow(x,\beta y)}{dy}\left(\frac{d\beta y}{dy}\right)^{-1}\right\}$$

* The difficulties which occur in treating functions with negative indices are similar to those in which they are positive; it may however be observed, that from the notation we have established, the following consequences follow:

$$\downarrow^{0,1}(x,y) = x \text{ and } \downarrow^{1,0}(x,y) = y$$

and generally $\downarrow^{0,n}(x,y) = x$ and $\downarrow^{n,0}(x,y) = y$

also $\downarrow^{0,0}(x,x) = x$ and if $\downarrow^{1,1}(x,y) = v$, then we have

$$x = \overline{\downarrow}^{1,1}(v,y) \text{ and also } = y\downarrow^{1,-1}(x,v)$$

and $\frac{d^2\psi(\alpha x, y)}{dxdy} = \frac{d^2\psi(x, \beta y)}{dxdy}$

hence $\frac{d^2\psi(x, \beta y)}{dx^2} = \frac{d\alpha x}{dx} \cdot \frac{d}{dy}\left\{\frac{d\psi(x, \beta y)}{dy}\left(\frac{d\beta y}{dy}\right)^{-1}\right\}$

put βy for y, observing that $\left(\frac{dy}{d\beta y}\right)^{-1} = \left\{\frac{dy}{dy}\left(\frac{d\beta y}{dy}\right)^{-1}\right\}^{-1} = \frac{d\beta y}{dy}$

and also $\frac{d\psi(x, y)}{d\beta y} = \frac{d\psi(x, y)}{dy}\left(\frac{d\beta y}{dy}\right)^{-1}$, then there will result the equation

$$\frac{d\beta y}{dy}\frac{d^2\psi(x, y)}{dx^2} = \frac{d\alpha x}{dx}\frac{d^2\psi(x, y)}{dy^2}$$

This is a partial differential equation from whose solution $\psi(x, y)$ may be found.

Ex. 1. Given the equation $\frac{d\psi(a-x, y)}{dy} = \frac{d\psi(x, b-y)}{dx}$ in this case $\alpha x = a - x$ and $\beta y = b - y$, and the differential equation to be solved is

$$\frac{d^2\psi(x, y)}{dx^2} = \frac{d^2\psi(x, y)}{dy^2}$$

and its solution is

$$\psi(x, y) = \varphi(x + y) + \varphi_1(x - y)$$

the two arbitrary functions φ and φ_1 must be determined so as to fulfil the given equation, for which purpose we have

$$\frac{d\psi(a-x, y)}{dy} = \varphi'(a - x + y) - \varphi_1'(a - x - y)$$

and $\frac{d\psi(x, b-y)}{dx} = \varphi'(b + x - y) + \varphi_1'(-b + x + y)$

φ' and φ_1' being respectively the differential coefficients of φ and φ_1, since these two expressions must be equal, we have

$$\varphi'(a - \overline{x - y}) = \varphi'(b + \overline{x - y})$$

and $$-\varphi_1'(a - \overline{x + y}) = \varphi_1'(-b + \overline{x + y})$$

whose solutions are

$$\phi'(x-y) = \chi\left\{\overline{x-y},\ \overline{a+b-x+y}\right\}$$

and $\phi_1'(x+y) = (a-b-2x-2y)\,\chi_1\left\{\overline{x+y},\ \overline{a-b-x-y}\right\}$

hence the general solution of the equation

$$\frac{d\psi(a-x,\,y)}{dy} = \frac{d\psi(x,\,b-y)}{dx}$$

is $\psi(x,y) = \int(dx+dy)\,\chi\left\{\overline{x+y},\ \overline{a+b-x-y}\right\} +$

$$+\int(dx-dy)\,(a-b-2x+2y)\,\chi_1\left\{\overline{x-y},\ \overline{a-b-x+y}\right\}$$

Ex. 2. Given the equation

$$\frac{d\psi\left(x,\,\frac{1}{y}\right)}{dx} = \frac{d\psi\left(\frac{1}{x},\,y\right)}{dy}$$

the partial differential equation to be solved is in this case

$$\frac{d^2\psi(x,y)}{dy^2} = \frac{x^2}{y^2}\,\frac{d^2\psi(x,y)}{dx^2}$$

and its solution is

$$\psi(x,y) = \left(\frac{x}{y}\right) + \phi_1(xy)$$

determining ϕ and ϕ_1 so as to fulfil the conditions of the equation, we have

$$\psi(x,y) = c\,(x+y) + \int d\,(xy)\,.\left(\frac{1}{xy}\right)^{\frac{1}{2}}\chi\left\{\overline{xy},\ \overline{\frac{1}{xy}}\right\}$$

Problem XLI.

Given the equation

$$F\left\{x, y, \psi(x,y), \psi(\alpha x, y), \&c.\ \frac{d^r\psi(x,\beta y)}{dx^r},\ \frac{d^s\psi(\alpha^s x, y)}{dy^s},\ \&c.\right\} = 0$$

and let $\alpha^p x = x$ and $\beta^q y = y$,

then there may be pq different forms of the function ψ contained in the general expression $\psi(\alpha^r x, \beta^s y)$, r varying from 0 to $p-1$, and s varying from 0 to $q-1$.

In the first place it may be observed, that if we substitute αx for x in such a quantity as

$$\frac{d^n\psi(\alpha^n x,\,\beta y)}{dx^n}$$

we shall have

$$\frac{d^n \psi(\alpha^n x, \beta y)}{(dx)^n}$$

which may always be reduced to the form

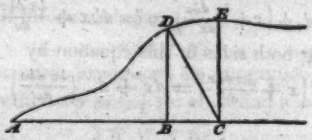
$$f(x) \frac{d^n \psi(\alpha^n x, \beta y)}{dx^n}$$

If now in the original equation we substitute successively αx, $\alpha^2 x$, . . $\alpha^{p-1} x$ for x, and βy, $\beta^2 y$, &c. $\beta^{q-1} y$ for y, we shall have pq equations containing pq forms of the unknown function and their differentials. By means of these pq equations and the differentials of them, we may eliminate all the different forms of the function ψ, except one: let the one which remains be $\psi(x,y)$, then we have an equation of partial differentials containing only x, y, $\psi(x,y)$ and their differentials: and from the solution of this equation $\psi(x,y)$ may be found; a certain number of arbitrary functions will be contained in this integral; these must all be determined so as to satisfy the original equation.

Amongst the numerous questions to which the calculus of functions is applicable, I shall select a problem proposed by EULER in one of the volumes of the Acta Acad. Petrop. as it will offer an example of a mode of treating of functional equations of a nature yet more general than those contained in this paper.

Problem XLII.

Required the nature of a curve such that taking any ordinate



DB, and drawing a normal at the point D, the next ordinate CE raised at the foot of the normal shall be equal to that normal.

Let AB $= x$, BD $= y$ and $y = \psi x$ be the equation of the curve, then BC $= \frac{y dy}{dx}$,

and DC $= \sqrt{y^2 + \left(\frac{y\,dy}{dx}\right)^2}$ and by the condition of the Problem we have

$$\sqrt{y^2 + \left(\frac{y dy}{dx}\right)^2} = \psi\left(x + \frac{y dy}{dx}\right)$$

hence

$$[\psi x]^2 + \left[\frac{\psi x\, d\psi x}{dx}\right]^2 = \left[\psi\left(x + \frac{\psi x\, d\ x}{dx}\right)\right]^2 \qquad (a)$$

This is apparently a very difficult functional equation, and I am not acquainted with any direct method of solving other similar ones. It is in fact only from a peculiar condition which this equation involves that any solutions have been obtained, the condition to which I allude is, that the quantity $\frac{\psi x\, d\psi x}{dx}$ does not change, when for x we substitute $x + \frac{\psi x\, d\psi x}{dx}$ or expressed in symbols, that

$$\frac{\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)d.\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)}{d\left(x+\frac{\psi x\, d\psi x}{dx}\right)}=\frac{\psi x\, d\psi x}{dx}$$

which may be thus proved differentiate (a) which gives

$$\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)d.\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)=\psi x\, d\psi x+\frac{\psi x\, d\psi x}{dx}\, d\,.\,\frac{\psi x\, d\psi x}{dx}$$

and by dividing both sides of this equation by

$$d\left(x+\frac{\psi x\, d\psi x}{dx}\right)=dx+d\,.\left(\frac{\psi x\, d\psi x}{dx}\right)$$

we have

$$\frac{\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)d\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)}{d\left(x+\frac{\psi x\, d\psi x}{dx}\right)}=\frac{\frac{\psi x\, d\psi x}{dx}\left(dx+d.\frac{\psi x\, d\psi x}{dx}\right)}{dx+d\left(\frac{\psi x\, d\psi x}{dx}\right)}=\frac{\psi x\, d\psi y}{dx}$$

From this it appears, that the subnormal is constant in the same series of ordinates, but it does not follow that it must be constant in different series; this property, viz. that $\frac{\psi x\, d\psi x}{dx}$ does not change when x becomes $x+\frac{\psi x\, d\psi x}{dx}$ will furnish us with a solution of the equation in question; for (a) becomes by putting t for $\frac{\psi x\, d\psi x}{dx}$.

$$[\psi(x+t)]^2-[\psi x]^2=t^2$$

where t may be considered as a constant quantity, the general solution of this equation is

$$\psi x=\sqrt{xt+\varphi t}$$

φt being an arbitrary function of t, therefore the general solution of eq. (a) is

$$\psi x=\sqrt{x\,\frac{\psi x\, d\psi x}{dx}+\varphi\left(\frac{\psi x\, d\psi x}{dx}\right)}$$

or

$$y^2=\frac{xy\, dy}{dx}+\varphi\left(\frac{y dy}{dx}\right)$$

from which differential equation the curves which satisfy the Problem may be found. It ought, however, to be observed,

that the constant quantity introduced by integration, is not perfectly arbitrary, it must be determined so as to make the equation between x and y fulfil the equation (a). If for instance, we assume $\varphi\left(\frac{ydy}{dx}\right)$ to be equal to $a\frac{ydy}{dx}$, we should find the equation of the curve to be

$$y=(a+x)c$$

c being the constant introduced by integration, and on substituting this value of y in (a) we shall find $c=o$, so that

$$y=(a+x)o$$

Let us suppose a to be infinite and equal to $\frac{b}{c}$, then we have

$$y=\left(\frac{b}{c}+x\right)c=b+cx=b, \text{ since } c=o$$

which is the equation of a straight line parallel to the axis of the x's, which in fact agrees with the conditions of the Problem. If we suppose $\varphi\left(\frac{ydy}{dx}\right)=a^2=$ a constant quantity, we should find

$$x=c\sqrt{y^2-a^2}$$

this value being substituted in (a) gives for determining c the equation

$$c^2(c^2+1)=o$$

whence $c=o$ and $c=\pm\sqrt{-1}$, using this latter value we have

$$x=\sqrt{-1}\times\sqrt{y^2-a^2}=\sqrt{a^2-y^2}$$

which is the equation of the circle, and it is obvious, that this curve satisfies the conditions.

It is very necessary to attend to this mode of determining the constants, as we should otherwise meet in the solution with many curves which do not satisfy the conditions; thus in the last example, the curve is apparently an hyperbola, but owing to the constant becoming imaginary, it is in fact a circle.

To complete the outline of this new method of calculation, it would be necessary to treat of equations involving two or more functional characteristics, and to explain methods of eliminating all but one of them: these lead to a variety of interesting and difficult enquiries, and will probably be of considerable use in completing the solutions of partial differential equations: it would also be proper to consider the maxima and minima of functions, and to apply to this subject the method of variations; these are points of considerable difficulty, and although I have made some little progress in each of them, I shall forbear for the present any farther discussion on this subject. In the mean time, the sketch which I have offered, and the few applications I have given, are sufficient to point out the great importance of this method. It should however be observed, that its applications have only been noticed incidentally; my object has been to direct the attention of the analyst to a new branch of the science, and to point out the manner of treating it: the doctrine of functions is of so general a nature, that it is applicable to every part of mathematical enquiry, and seems eminently qualified to reduce into one regular and uniform system the diversified methods and scattered artifices of the modern analysis; from its comprehensive nature, it is fitted for the systematic arrangement of the science, and from the new and singular relations which it expresses, it is admirably adapted for farther improvements and discoveries.

XII. *Experiments and observations to prove that the beneficial effects of many medicines are produced through the medium of the circulating blood, more particularly that of the colchicum autumnale upon the gout. By Sir* Everard Home, *Bart. V. P. R. S. Communicated by the Society for improving animal chemistry.*

Read March 21, 1816.

A KNOWLEDGE of the readiness with which liquids pass from the stomach into the circulation, carrying along with them the impregnation of different medicines; and the readiness with which such medicines are carried off from the circulating blood, by the action of the kidneys, led Mr. BRANDE and myself to an enquiry respecting the prevention of gravel and gout, upon which subject he has laid two separate papers before the Society.

In these communications, the action of different substances on the contents of the stomach has been considered, and those substances most efficient in depriving them of the principal ingredient met with in stone and gout, are pointed out.

For the cure of gout, the eau medicinale of Husson has been most fortunately discovered to be a specific remedy, and it is now ascertained, by experiments on different people, that a vinous infusion of the colchicum autumnale, or meadow saffron, is equally so, and therefore the two medicines must be considered as the same.

To ascertain their mode of action, appeared to me an enquiry connected with the objects of this Society, which are not con-

fined to the knowledge of purely chemical combinations in the stomach, or other parts of the body, but include the effects of galvanism on the nerves, and of mineral and vegetable solutions on the blood, so far as they affect the actions of life, or the symptoms of disease.

It has already been determined by experiment, that almost every mineral, vegetable, and animal poison, if not the whole of them, is carried into the circulation before it produces its specific effects upon particular parts, whether these are the stomach, skin, or other parts of the body. The most truly specific medicine that we have been hitherto acquainted with, is mercury for the venereal disease, and it is completely established, that this remedy, when in the circulation, is equally efficient in the cure of a recent chancre produced by inoculation, and a venereal sore throat, in consequence of the disease having been carried into the circulation.

That other medicines can be received into the circulation, and, as soon as they arrive there, produce their effects upon different parts of the body, is proved by experiments made by the late Mr. Hunter, although he had no idea of their being usually carried there before they produce the different actions so well known to follow their exhibition by the mouth. He found that infusions of the following substances received into the circulation by the jugular vein, immediately produced the same effects which more slowly follow their being taken by the mouth. Infusion of opium brought on drowsiness. Infusion of ipecacuanha vomiting. Jalap vomiting and purging. Infusion of rhubarb a profuse flow of urine. These effects ceased in a few hours, and appeared to have in no respect injured the animal's health. Except the

venereal disease, gout is the only one whose local symptoms have been completely removed by medicine, in so short a time, as to put it beyond all doubt that their removal is the effect of the medicine. The effect of the eau medicinale and of the vinous infusion of the colchicum autumnale on gout, is indeed more rapid than that of mercury on the venereal disease, but in all other respects corresponds with it, and if these medicines act through the medium of the circulation, the only difference may be, that the one is more quickly received into it than the other.

This power of the eau medicinale, which I have stated to be exactly similar to that of the colchicum autumnale over the local symptoms of gout, I have ascertained by experiment more than six times upon myself; at one time the symptoms went off in six hours, at another in 12, and at others in 24 hours.

As we know the sensible effects of mercury, whether it is introduced into the circulation by the absorbents, or received into the stomach, are the same, we conclude whenever these sensible effects are met with, that mercury is actually in the circulation.

It therefore occurred to me, that if the sensible effects of the infusion of the colchicum should prove to be the same, whether it is introduced into the circulation by the jugular vein, or received by the mouth into the stomach, that we might equally in both cases conclude it to be in the circulation. To determine this point, 80 drops of the vinous infusion of the colchicum (made by macerating two pounds of the fresh roots in 24 ounces of Sherry wine, in a gentle heat for six days, the spirit being previously carried off by heat,) was diluted

with a dram of water, and conveyed into the circulation of a moderately sized dog by the jugular vein. The dog's pulse in a natural state is 140 in a minute.

In 5 minutes, the dog had a tremulous motion of the muscles and fluttering of the pulse, accompanied with nausea, but no retching to vomit. In 14 minutes, the pulse was 180 in a minute and had frequent intermissions. In 4 hours, the pulse was 120 in a minute, of its natural strength, and had frequent intermissions. In 7 hours, the dog had a natural motion, the pulse had no intermission, was 140 in the minute. The dog had a good appetite for food, and appeared in perfect health.

The same dog at the end of three complete days, swallowed 60 drops of the same infusion, exactly double the quantity that had been introduced into the circulation. In 2 hours, he became languid, the pulse wiry and weak, but 140 in the minute. In 4½ hours, the languor much less and the pulse natural. In 8 hours, the dog had had a natural motion. In 11 hours, was in good spirits and very well.

The sensible effects, upon the dog, were similar to those produced upon myself, but in a less degree. Under the influence of a violent fit of the gout, in the ankle, on the 23d of December, 1815, at 10 o'clock in the morning, I took 60 drops of the eau medicinale; the pain of the gout was insufferable, I got into bed, and was so chilly as not to be able to keep my hands warm, even under the bed clothes. In 2 hours, I became rather hot and thirsty. In 3 hours, the pain was so much diminished as to be tolerable, while the limb was at rest. In 7 hours, I had a confined motion from the bowels, and the pain in the ankle became severe, while the foot was placed

on the ground, but this went off as soon as the foot was again placed in a horizontal posture. A nausea, or half sickness, came on; my pulse, which is naturally 80 in a minute, was lowered to 60, and intermitted. In 10 hours, the nausea was gone off, but I remained languid, the pulse beating 70 in a minute. I had some appetite for food.

The following morning, my pulse was 80, and having passed a good night, I was enabled to walk as usual, and follow the duties of my profession.

If these observations shall be confirmed, they must lead us to conclude, that the different kinds of substances, which produce specific diseases, are first carried into the circulation, in the same manner as mineral and animal poisons, and that the medicines by which they are acted upon, go through the same course, before they produce their beneficial effects; a material step will thus be gained in the consideration of diseases, and the modes of treating them.

XIII. *An appendix to a paper on the effects of the colchicum autumnale on gout.* By Sir Everard Home, *Bart. V.P.R.S.*

Read April 25, 1816.

When I laid before the Society my Paper upon this subject, I was anxious to establish what appeared to me to be two important facts; one, that the infusion of the colchicum can be received into the circulation without producing any permanent mischief; the other, that it is through the medium of the circulation, its beneficial effects upon gout are produced, and, therefore, the sudden relief which is experienced can be readily explained. Having attended to the effects of the eau medicinale, and of this medicine for several years in cases of gout, both in my own case, and in those of my friends, I found, invariably, that they diminished the frequency of the pulse, 10 or 20 beats in a minute, and this effect generally took place about twelve hours after the medicine was exhibited: I therefore considered this to be the criterion of the constitution being under the influence of the medicine; and when I found that the pulse was affected in the same way by the medicine received into the circulation, and in a much shorter time, I became satisfied that in both cases this arose from an effect upon the circulation, and not upon the stomach, and therefore did not farther prosecute the enquiry; since exhibiting larger doses could only confirm what is already known, namely, that the medicine is capable, when injudiciously used, of producing very violent effects.

It has been suggested to me since the Paper was read, that the only mode of proving that the medicine acts through the medium of the circulation, is to show that when a sufficient quantity is received into the blood, all the violent effects are produced, that result from a large dose taken by the mouth; and as I had no object but the pursuit of truth, I lost no time in complying with this suggestion, and introduced into the circulation of a dog 160 drops of the same infusion before employed.

The animal instantly lost all power of voluntary motion, the breathing became extremely slow, and the pulse was hardly to be felt. In 10 minutes, the pulse was 84, the inspirations natural, which are 40 in a minute. In 20 minutes, the pulse was 60, the inspirations 30 in a minute, a tremulous motion had taken place in the hind legs. In an hour, the pulse was 115, and irregular; the animal was capable of sitting up, but was in a state of violent tremor, and the inspirations could not be counted.

In 1½ hour, the tremor had gone off, the pulse continued the same; the animal made ineffectual attempts to vomit, and continued to do so for ten minutes, accompanied with great languor; the inspirations were 54 in a minute.

In 2 hours, the pulse was 150, and very weak; the animal had voided 1½ ounce of water, had vomited twice, each time bringing up a quantity of mucus tinged with bile, and had two liquid stools.

In 3 hours, had vomited again, and had another stool; the pulse too weak to be counted.

In 4 hours, continued extremely languid.

In 5 hours, vomited some bloody mucus, and expired.

On opening the body, the stomach contained mucus tinged with blood, and its internal membrane was inflamed; the duodenum had its internal surface universally inflamed, the same appearance in a less degree was met with in the jejunium and ilium, and more strongly marked in the colon than in the ilium.

The facts which I have now adduced, afford sufficient proof of the action of the colchicum autumnale upon the different parts of the body, being through the medium of the circulation, and not in consequence of its immediate effects upon the stomach and intestines.

XIV. *On the cutting diamond.* By W. H. Wollaston, *M. D. Sec. R. S.*

Read May 9, 1816.

When we consider how long the diamond has been in common use for the purpose of cutting glass, it is rather surprising that no adequate explanation has been given of that remarkable property, and that even the conditions on which the effect depends have not been duly investigated.

Many persons, indeed, are not aware of the distinction that is to be drawn between scratching and cutting. In the former, the surface is irregularly torn into a rough furrow; in the latter a smooth fissure, or superficial crack, is made, which should be continued without interruption from one end to the other of the line in which the glass is intended to be cut. The skilful workman then applies a small force solely at one extremity of this line, and the crack which he forms is led by the fissure almost with certainty to the other.

Any other substance harder than glass, possesses the power of *scratching* in common with the diamond. But the power of *cutting* has been thought confined to the diamond; and it is true that its peculiar hardness certainly contributes to the duration of that power.

I was informed that persons employed in setting diamonds for the use of the glazier, always select natural diamonds distinctly crystallized, which they term sparks; but upon what

circumstance this supposed superiority of the natural diamond over that which has been cut by art, could depend, I was not able to gain any information.

Having procured a common glazier's diamond ready set, and such a quantity of glass as I thought would be sufficient for learning by experiment the art of cutting, I endeavoured first by forcible pressure on the point in different directions to effect my purpose. But although I could thus tear the surface to a considerable depth, I could by no means command the direction of the fracture.

When I placed the diamond more inclined to the surface, I could occasionally, and in part, obtain what I thought to be a proper cut; but I was unable to continue the stroke with steadiness, and so incapable of repeating it a second time with a similar effect, that I was convinced the precise direction necessary for cutting was confined within very narrow limits.

Having found that the diamond required to be moved in the direction of one of its edges, and having by repeated trials formed a judgment of the requisite inclination of its handle, I mounted it in a frame, in which I could fix it at any angle of elevation that appeared suitable, and could turn it round its axis to adjust the direction of its edge. By this arrangement I had no difficulty in repeating any successful trial, or of varying it according to hints derived from such imperfections as were observable; and I soon discovered that difference in the form of the natural diamond, from that of diamonds cut by art, on which I believe the power of cutting to depend.

When a diamond is formed and polished by the lapidary, all the surfaces are *plain* surfaces, as far as it is in his power to make them so, and consequently the edge or line in which

they meet is straight. But in the natural diamond there is this peculiarity in those modifications of its crystals that are chosen for this purpose, that the surfaces are in general all *curved*, and consequently the meeting of any two of them presents a curvilinear edge. If the diamond be so placed, that the line of the intended cut is a tangent to this edge near to its extremity, and if the two surfaces of the diamond laterally adjacent be equally inclined to the surface of the glass, then the conditions necessary for effecting the cut are complied with. The curvature, however, of the edge is not considerable, and consequently the limits of inclination are very confined; for if the handle be either too much or too little elevated, then one or other extremity of the curve will be made to bear angularly upon the glass, and will plough a ragged groove by pressure of its point. But on the contrary, when the contact is duly formed, a simple fissure is effected as if by lateral pressure of the adjacent surfaces of the diamond directed equally to each side. By that means, adjacent portions at the surface of the glass are forced asunder farther than the mere elasticity of the parts beneath will allow, and a partial separation or superficial crack is produced.

The effects of inequality in the lateral inclination of the faces of the diamond to the surface of the glass, are different according to the degree of inequality. If the difference be very small the cut may still be clean; but as the fissure is then not at right angles to the surface, the subsequent fracture is found inclined accordingly. But when an attempt is made to cut with an inclination that deviates still more from the perpendicular, the glass is found superficially flawed out

on that side to which the greater pressure was directed, and the cut completely fails.

It might be thought that the weakness of the glass in this part would nevertheless occasion it to break in the desired direction; but the bottom of a flaw is in fact of very great breadth when compared to the simple crack produced in a proper cut. In one case the force applied to break the glass is dispersed over a space of some extent, and may be diverted from its course; in the other the whole force is confined successively to the mere points of a mathematical line, which may be conceived the bottom of the fissure, and is directed onward by the facility with which the adhesion of each particle in succession yields to its progress.

The depth to which the fissure made by the diamond penetrates, need not be greater than $\frac{1}{200}$ of an inch, for I found that the fracture might be completely turned from its course, at any part of the intended line, by grinding away a portion of the surface; and by an average of several experiments the thickness of the glass was not found to be diminished so much as $\frac{6}{1000}$ths of an inch.

Since the form of the cutting edge appeared from the above trials to be the principal circumstance on which the property of cutting depends, I thought it not improbable that other stones possessed of the requisite hardness, might be found to produce the same effect, if brought to a similar curvilinear edge. By a little pains I succeeded in giving this form to a sapphire, a ruby, a spinell ruby, to rock crystal, and some other substances, and found that each of these bodies has the power of cutting glass for a short time with

a clean fissure. But notwithstanding the hardness of the ruby was such as to occasion a great deal of labour in giving it the form I wished, the edge of this stone was by no means proportionally lasting. I am inclined to ascribe this defect in part to the grain or position of its laminæ having been unluckily oblique. And it seems highly probable that the singular durability of the edge of the cutting diamond, is owing in some measure to this circumstance, that its hardness in the direction of the natural angle of its crystal, is greater than in any other direction, as we find to be the case in other crystals of which the various degrees of hardness in different directions can be more easily examined.

XV. *An account of the discovery of a mass of native iron in Brasil. By* A. F. Mornay, *Esq. in a letter to* W. H. Wollaston, *M. D. Sec. R. S.*

Read May 16, 1816.

DEAR SIR,

NEAR five years have elapsed since I presented you with a specimen of native iron from Brasil. Particular reasons prevented me at that time, from making it more generally known, and since then my private affairs have not allowed me a moment to look into my notes, and give you this short account of the block from which your specimen was cut, although I have so often promised it you.

In the autumn of 1810, I discovered near Bahia, a spring of water strongly impregnated with iron, which was esteemed a most valuable acquisition in that country. This circumstance called to the recollection of the government, that, about 30 years before, information had been received of the discovery of certain thermal springs, situated at the distance of 40 or 50 leagues to the northward; and as his Royal Highness the Prince Regent of Portugal had enquired, during his stay at Bahia, whether the country possessed any thermal waters, I was requested to visit the spot where they were supposed to exist. The Governor General offered me every facility and protection, and in order to induce me to undertake the journey, some of my friends described to me an

extraordinary *stone* which had been found still farther up the country, in the same direction. It had been supposed to be silver, or iron, or that ferruginous agglomeration so common in Brasil, which often envelopes gold, and I believe sometimes diamonds. On the other hand, some persons who pretended to have seen it, asserted that it was not a mass of any metal, but had only the metallic sound on being struck, common to numerous blocks of stone in the same neighbourhood, called by the inhabitants "serpent stones," in consequence of their exfoliating by decomposition at the surface. As the serpent casts his skin yearly, so they suppose these stones to do.

Some account of the discovery of this extraordinary mass had been given to the government of Bahia, and through the inspector general of the militia, a man of great talents and considerable learning, I obtained a sight of the papers on the subject existing at the government house. On reading them, I was decidedly of opinion, that the mass described was native or meteoric iron, and I determined to go to see it. But before I relate my own observations, I will give you the substance of the notes which I took out of those papers.

In the year 1784, a man of the name of BERNARDINO DA MOTA BOTELHO, while looking after his cattle, noticed the block in question, as being different from all the other stones on the spot, and informed the Governor General of the province of Bahia of his observation. His Excellency immediately ordered the head man of a neighbouring village, that is to say, at the distance of near fifty leagues, to go and examine it. He did so, and reported very marvellous things, calling the mass sometimes iron, and sometimes stone, but giving to understand that it contained gold and silver. The Governor

General commanded him, in consequence, to have it conveyed to Bahia. This man returned to the spot, and after having excavated round the block, so as to be able to get the ends of four powerful levers under it, he contrived by great exertion, with the assistance of thirty men, to turn it on its side. He observed the bed on which it rested, to be of the same scaly substance that was attached to the bottom of the mass, and about eighteen inches thick.

About the latter end of 1785, he conveyed to the spot a waggon, or rather a truck built for the purpose, and succeeded in getting the mass of iron into it, but having spent three days in this operation, the men employed were obliged to depart, in consequence of the neighbouring rivulet being brackish, and not fit to be drank. They returned, however, and yoked oxen to the truck, but they could not move it until they had put on twenty pair of oxen on each side. You must observe that their oxen are not of the strength of ours, that the ground was a loose gravel, and that the truck was constructed on the very worst plan, the wheels being fixed to the axle trees, and the two axle trees remaining constantly in a parallel position with respect to each other.

They proceeded, however, in this manner to the distance of about one hundred yards, when they got into the bed of the rivulet abovementioned, called the Bendegó. There it was stopped by the prominent point of a rock, and as the truck was only calculated to move in a straight line, it was abandoned.

I visited this mass on the 17th of January, 1811, and found it still on the waggon or truck, where it had been lying for five and twenty years. It is situated near the left bank of

the rivulet, but entirely in its bed, which was then dry, and is very seldom otherwise.

I send you a very correct outline of this mass. (Pl. XI.) It is about 7 feet long, 4 feet wide, and 2 feet in thickness, besides a sort of foot on which it now stands, of about six inches in height. The solid contents, however, cannot be inferred correctly from these dimensions, since the broad part is hollowed out underneath very considerably. After making due allowance for the cavities, I estimated, on the spot, the solid contents of the whole mass to be at least 28 cubic feet, which at 500lb. will make its weight to be 14000lb.

Its colour is exactly that of a chesnut, and is glossy at the top and sides, but the hollow part underneath is covered with a crust in thick flakes, outwardly of the colour of rust of iron, and staining the fingers. The flakes are very brittle, and the fresh fracture is black and brilliant, like some magnetic iron ores.

The glossy surfaces of the block are not smooth, but slightly indented all over, as if they had been hammered with a rather large round headed hammer.

There are several cavities in it, from the diameter of a 12lb. cannon ball, to that of a musket ball; the larger ones being shallow, but the others much deeper. They all contain the same substance as is attached to the great cavity underneath, and some of them also fragments of quartzous stones, which I was obliged to break in the holes in order to get them out.

The brown colour of the surface of the block is merely a very thin coat of rust, for the slightest scratch with a knife produces a bright white metallic streak; and yet,

wherever the mass is struck with a steel, it gives out sparks abundantly.

When rubbed with a quartzous pebble in the dark, it becomes beautifully luminous.

The block is magnetic, and even possesses well marked poles. In the outline I have indicated their position. The N. pole is not so well characterized at the shorter point of the same end.

The N. pole of the block lies at present nearly E.N.E.: before it was removed it lay about N.N.E. I ought to tell you that LA MOTA BOTELHO, who first noticed this object, accompanied me, and, as he was present at its removal, he was able to give me much information, being a very intelligent man.

The N. pole is by much the most massive end, and lay deeper in the ground than the other.

No part of the mass has the power of attracting iron filings, whether the spot have been filed to brightness or not.

I had provided myself with a sledge hammer and tools for cutting off some specimens of the iron, but it was with the utmost difficulty that I could detach the few small pieces which you have seen, one of which I gave to you on my arrival in England. The largest I presented to my Lord DUNDAS, to whom I am under many obligations, and who promised to place it in the collection of the Geological Society. I also presented fragments to our lamented friend Mr. TENNANT, and to Dr. MARCET. Another specimen, beautifully crystallized, I disposed of to Mr. HEULAND, and I have only some small pieces left. As soon as the first piece was detached, I was struck with the appearance of internal crystallization not

hitherto noticed in meteoric iron, but as your specimen shows this circumstance very well, I need not describe it.

None of the fragments possess magnetic poles.

No vitreous substance appears about the mass, as in many of the known blocks of meteoric iron.

Having taken a few reagents with me, for the examination of the thermal springs which had been pointed out to me, I tried the malleable part of the mass on the spot, for nickel, and I thought at the time that its presence was indicated; but I am now satisfied that the phænomena which I noticed, might have arisen from iron alone.

I have found my specimens more liable to rust, I think, than wrought iron generally is; and in a damp atmosphere a liquid oozes out from the crevices.

I repaired to the spot where the mass was discovered, namely, on a rising ground on the left bank of the river Bendegó, and caused the soil and gravel to be removed until we came to the bed described in the government documents. We found it at less than three feet depth. I had expected to find in it a considerable protuberance, such as might have fitted the cavity underneath the mass of iron, for I was convinced that the block itself must have been firmly attached to the bed, otherwise it would not have required such a considerable power to turn it on its side.

However, I did not; and thinking that we were not exactly on the spot, I caused two trenches to be opened down to the bed, and crossing each other, the one being between two and three yards long, and the other between one and two. Every part of the bed that was uncovered was perfectly flat and horizontal, except where we dug first; there it was broken, and,

according to the statement of LA MOTA BOTELHO, that was done when the block was removed.

I found no termination to the bed in the directions of the trenches, and at the spot where the mass had laid, it was about one foot thick, or hardly so much; but at one end of the longer trench, not above three inches. I did not break through it any where else. Nearly the same loose gravel appears underneath the bed, as over it. I brought away specimens of the bed, which I considered extremely curious, supposing them to contain nickel. On my return to England I told you, therefore, that I hoped I had found iron ore containing nickel, for I thought that the bed, on which had rested the mass, was one of those of which there are so many all over the province. But as I gave you some specimens, I will not describe it.

The surface of the soil, or rather coarse gravel, at the spot, is about 10 or 15 feet above the main granite rock of the country.

I can only give you an approximation of the latitude and longitude of the place. Th sun was much too high at noon to take its altitude with a sextant and mercurial horizon; and the artificial horizon, which I had been compelled to construct myself, occasioned such a loss of light, as to make it impossible to observe the southern stars for determining the latitude. Different altitudes of the sun at a distance from the meridian, did not give me satisfactory results. I had with me an excellent watch, and having computed the latitude to be about 10° 20′ S. I concluded the longitude to be 33′ 15″ W. of Bahia, after making every allowance, and comparing this result with those obtained before and afterwards, at the house of Major DANTAS, called Camuciatá, near Itapicurú.

The rapidity of growth in plants is wonderful in the neighbourhood of the Bendegó, although the main granite rock is so near the surface as to protrude in many places; and what lies on it is chiefly a coarse gravel, consisting of rolled fragments of quartz, felspar and granite of the size of eggs, together with smaller pebbles and sand, which contains, of course, a great deal of mica, but hardly any vegetable earth.

At about 40 leagues to the southward, are found hills of yellow and red sand stone, in which organic remains have not been found; while to the northward, there is a formation of similar hills, in which are observed most beautiful impressions of whole fishes and remains of vegetables.

Between the Bendegó and the sandstone hills to the southward, I observed a great deal of what I certainly take to be basalt. I met with balls from the diameter of two inches to that of upwards of three feet, and numberless prisms, with three and with six faces, scattered about; all of these small, that is to say, about three or four inches in length, and two or three in diameter.

To the southward of the sandstone hills is a sandy plain, almost barren, extending many miles, perhaps 60 or 80, east and west to the sea, but not 20 in breadth, where I crossed it. Small conical hillocks are scattered over it, some of which, the largest, have flat tops, and appear all to be of the same height, about 20 fathoms.

Appearances impressed me with the idea, that they were the remains of a plain which formerly extended over the one on which I then stood, but which had been washed away in a tumultuous manner by a violent current running nearly in an easterly direction. The larger hillocks appear to be

stratified, but they consist of loose sandy materials, except in so far as they contain beds of a dark red iron ore, containing imbedded minute crystals of magnetic iron ore: the thickness of these beds is about two inches, and they are exactly similar to those which are found in the clay hills of Bahia.

The smaller hillocks consist of confused heaps of gravel and loose stones, intermixed with a very large quantity of the same iron ore in fragments, and lumps of manganese, very compact, and of a steel grey colour, containing arsenic, but apparently no iron.

The dreary appearance of this plain is increased by the numerous nests of cupim, (white ants,) standing upright like so many tombstones. On being viewed nearer, they are conical, rather compressed, so that the base is elliptic. All those which I examined were precisely of the same shape. The materials of which they consist are white sand, whitish clay, and particles of wood.

Many of them were full five feet in height.

The soil of the valleys and low grounds, which are occasionally swampy, is abundantly impregnated with sea salt, which the inhabitants wash out for their own consumption; but it contains some bitter salts, which render it purgative to those who are not accustomed to it.

The thermal springs which were pointed out to me, were several, but they hardly deserve the name.

One of them was at 86° of Fahrenheit when the atmosphere was at 81°.

Another was at 88°, when the atmosphere was at 77½°; and also at 88° when the atmosphere was at 80°.

The water of both of these is the purest I had ever seen.

Many small fish were swimming in the basin of the last, from which runs, at all seasons, a considerable rivulet.

A third was at 90° when the atmosphere was at 78°. The water very pure.

A fourth was at 101° when the atmosphere was at 85½°; also at 101° when the atmosphere was at 98°.

Taste of the water rather ferruginous, and very brackish, extremely disagreeable and nauseous. No peculiar smell, and very transparent, although it deposits iron and lime, and an iridescent film is formed on its surface. Contains no sulphuretted gas. The rocks of the neighbourhood contain pyrites not magnetic.

This spring is called the Mai-d'agoa, and is situated on the left bank of the river Itapicurú, near the water's edge, at a short distance from a place called the Mato-do-cipó.

It was during this journey that I had an opportunity of seeing that curious plant called cipó de cunanam. It grows abundantly between Monte Santo and the river Bendegó. It is a climbing plant destitute of leaves; it was so when I saw it, and I believe it to be always the same; it bears no thorns; but often growing so as to form an impenetrable *plica* which the cattle will hardly approach, much less attempt to break through, because when the juice of this plant sticks to their hair, it occasions blisters and great irritation. It contains a milky juice, and I suppose that it is an euphorbium. When I made a cut at the bush with my hanger, in the dusk of the evening, the wounds inflicted presented a beautifully luminous line, which was not transient, but lasted for several seconds, or a quarter of a minute. Having taken a piece of the plant, I bent it in the dark until the skin cracked, when every

crack showed the same light, which is of a phosphorescent appearance. I continued to bend the twig until the milky juice dropped out, when each drop was a drop of fire, very much like what I have seen on dropping inflamed tallow. I did not observe any particular smell. The milky juice is said to be very poisonous; it is caustic, and occasions much itching and irritation when applied to the skin. It becomes viscous in the air, and soon dries of a yellowish colour, slightly tinged with green, when it has the appearance of a gum-resin.

The above account contains all the information that I can give you on the subject: should you think it deserving to be laid before the Royal Society, I would beg of you to add your observations, as they would render the communication interesting.

I am, with sentiments of the highest esteem and respect,

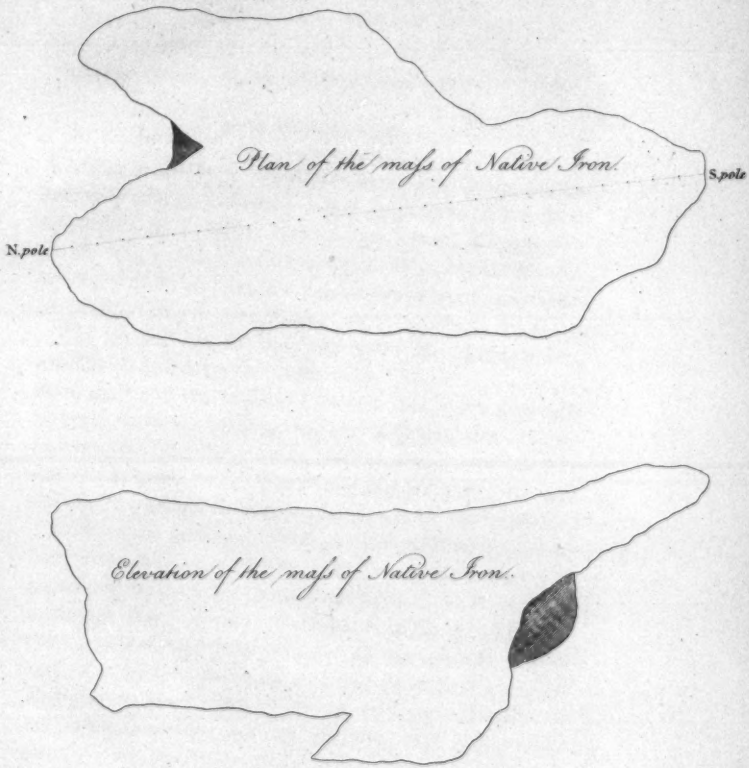
My dear Sir,

Your faithful friend and devoted servant,

A. F. MORNAY.

London, 27th April, 1816.

To Dr. Wollaston,
Secretary of the Royal Society.


Plan of the mass of Native Iron.
S. pole
N. pole
Elevation of the mass of Native Iron.

XVI. *Observations and experiments on the mass of native iron found in Brasil.* By W. H. Wollaston, *M. D. Sec. R. S.*

Read May 16, 1816.

THE preceding letter from Mr. MORNAY, relating to the discovery of a mass of native iron in Bahia, was drawn up at my request, as a valuable addition to our stock of knowledge on that most curious subject; and I am in hopes that the results of my own experiments may contribute something not uninteresting to the Society.

The specimen of the iron with which Mr. MORNAY very liberally supplied me for experiment, though it necessarily bears marks of the hammer by which it has been detached, presents also other surfaces, not only indicating that its texture is crystalline, but showing also the forms in which it is disposed to break, to be those of the regular octohedron and tetrahedron, or rhomboid, consisting of these forms combined.

In my own specimen, the crystalline surfaces appear to have been the result of a process of oxidation, which has penetrated the mass to a considerable depth in the direction of its laminæ; but in the specimen which is in the possession of the Geological Society, the brilliant surfaces that have been occasioned by forcible separation from the original mass, exhibit also the same configurations, as are usual in the fracture of octohedral crystals, and are found in many simple native metals.

The magnetic qualities of the fragments, fortunately,

enable us to appreciate rightly, those of the entire mass from which they have been detached; for though the mass, when tried upon the spot by Mr. MORNAY, gave indications of having distinct N. and S. poles, it is pretty clear that these were only so by induction, in consequence of position with respect to the magnetic meridian. For though the fragments are not in the least attractive as magnets, and have in themselves no polarity, they are precisely like any other pieces of the best soft iron, and assume polarity instantly, according to the position in which they are held with respect to the magnetic axis of the earth. When a long fragment is held in a vertical position, its lower extremity being then within 20° of the dip of the N. magnetic pole, becomes N., and repels the N. pole of a magnetic needle suspended horizontally. But this power is instantly reversed by being suddenly inverted. So that the apparent contradiction between the observed polarity of the mass, and the seeming want of it in the fragments, is thus completely removed.

Although Mr. MORNAY reasonably expected that this iron would not differ from the many others now on record that have been found in various parts of the world, and from his experiments was led to infer the presence of nickel, it appeared desirable to ascertain this point with more precision than he had been enabled to do, and to determine also in what proportion this peculiar ingredient of meteoric bodies might be found to prevail.

I believe the means which I am accustomed to employ for detecting the *presence* of nickel in native iron to be new, and may deserve to be described, on account of the very small quantity of the iron required for this mode of examination.

Having filed from my specimen as much as I judged sufficient for my purpose, (which need not exceed $\frac{1}{100}$ of a grain), I dissolved it in a drop of nitric acid, and then evaporated the solution to dryness. A drop or two of pure ammonia was then added to the dried residuum, and gently warmed upon it in order to dissolve any nickel that might be present. The transparent part of the fluid was then led by the end of a rod of glass to a small distance from the remaining oxide of iron, and the addition of triple prussiate of potash immediately detected the presence of nickel by the appearance of a milky cloud, which was not discernible by the same means from a similar quantity of common wrought iron tried at the same time.

For the determination of the *quantity* of nickel I employed a different method. A piece of the iron weighing 50 grains having been dissolved in nitro-muriatic acid, the solution was evaporated to dryness. Ammonia was then added, and the solution again evaporated to dryness, in order that the oxide of iron might be rendered more dense, and more easily separated from the soluble portion. A fresh addition of ammonia then readily dissolved the nickel, and the solution after filtration appeared of a deep blue colour.

A small quantity of sulphuric acid having then been added, the whole was again evaporated not merely to dryness, but with sufficient heat to expel the excess of ammonia, muriate of ammonia, and sulphate of ammonia. The remainder was sulphate of nickel which was then redissolved in water, and after being suffered to crystallize weighed 8,6 grains. Having found by experiment previously made for that purpose,

that 10 grains of nickel give 44 grains of sulphate of nickel, I infer that 8,6 of the sulphate correspond to 1,95 of metallic nickel, which is nearly 4 per cent of the quantity of native iron taken for experiment.

By an analysis conducted in a similar manner on 23 grains of the scaly flakes of oxide brought home by Mr. MORNAY, from the spot were the mass was found, I obtained 3,1 grains of sulphate of nickel, which correspond to 7,05 nickel, amounting to no more than 3,06 per cent. of the oxidated crust taken for analysis. But, if we consider the weight which 100 parts of the metallic alloy would acquire by oxidation, we shall find the two experiments correspond with a degree of accuracy that may occasion more reliance to be placed on these experiments than they really deserve.

96 parts of iron in the state of black oxide will be combined with 28,3 oxygen
and 4 nickel will take
about 1.1 oxygen,

so that 129,4 of the crust will contain only 4 parts of metallic nickel, and 100 ditto will contain 3,1, which scarcely exceeds the quantity actually found by trial.

From the presence of nickel in this mass we cannot but regard it as having the same meteoric origin with the various other specimens that have before been found; and although in the spot whence it had been first removed, Mr. MORNAY discovered a bed of matter from which it appears, by analysis, that similar iron might be formed by art, it seems by far more probable, that an opposite change has really taken place, and that the whole of this supposed ore is the result of pro-

gressive oxidation, during a series of years of which we have no other evidence, and affords the sole ground on which a conjecture could be formed of the very remote period at which this problematic body has fallen upon the earth.

XVII. *On ice found in the bottoms of rivers. By* T. A. Knight, *Esq. F. R. S. In a letter addressed to the Right Hon. Sir Joseph Banks, Bart. G. C. B. P. R. S.*

Read May 23, 1816.

DEAR SIR,

ACCOUNTS of ice having been found in the bottoms of rivers, have been so numerous, that the existence of it, in such situations, has ceased to be questioned, though no satisfactory hypothesis, relative to the manner in which it can be formed in, or conveyed to, such situations has, I believe, ever been offered. Indeed its existence in such situations does not appear to be a common occurrence, and possibly it has never been seen, during its progressive formation, by any person likely to trouble himself about the causes of its existence. I therefore take the liberty to submit the following account to your consideration.

I first witnessed the existence of ice in the bottom of the water in the river Teme, which passes near my residence in Herefordshire, in the last winter. In a morning which succeeded an intensely cold night, the stones in the rocky bed of the river, appeared to be covered over with frozen matter, which reflected a kind of silvery whiteness, and which, upon examination, I found to consist of numerous frozen spicula crossing each other in every direction, as in snow; but not having any where, except very near the shore, assumed the

state of firm compact ice. The river was not, at this time, frozen over in any part; but the temperature of the water was obviously at the freezing point, for small pieces of ice had every where formed upon it in its more stagnant parts near the shores; and upon a mill pond, just above the shallow streams, (in the bottom of which I had observed the ice,) I noticed millions of little frozen spicula floating upon the water. At the end of this mill pond, the water fell over a low weir, and entered a narrow channel, where its course was obstructed by points of rock and large stones. By these, numerous eddies and gyrations were occasioned, which apparently drew the floating spicula under water; and I found the frozen matter to accumulate much more abundantly upon such parts of the stones as stood opposed to the current, where that was not very rapid, below the little falls, or very rapid parts of the river. I have reason to believe, that it would have accumulated in very large quantities, if the weather had continued sufficiently cold; for I had previously heard, from persons of respectable character, who had no interest, nor, I believe, intention, to deceive me, that, during a long and severe frost, some years ago, before I became an inhabitant of my present house, the whole bed of the river, in the part above-mentioned, had been covered over with a thick coat of ice. But it was not till the month of February that I witnessed the apparent deposition of ice in the manner which I have described; and as the day afterwards became bright, the spicula soon ceased to form, and the ice to accumulate; and before the middle of the day the greater part of it had disappeared.

Upon some large stones, near the shore, of which parts

were out of the water, and upon pieces of native rock, under similar circumstances, the ice beneath the water had acquired a firmer texture, but appeared from its whiteness, to have been first formed of congregated spicula, and to have subsequently frozen into a firm mass, owing to the lower temperature of the stone, or rock. Ice of this kind extended, in a few places, eighteen inches from the shore, and lay three or four inches below the level of the surface of the water, and did not dissolve nearly so rapidly as that which was deposited upon stones more distant from the shores.

The cause of the appearance of large quantities of porous ice, in some of the continental rivers, upon a thaw taking place after a long and severe frost, may, I conceive, be explained, without much difficulty, consistently with the foregoing hypothesis: for such ice would be removed by the increasing force of the rising water, and might be driven together in large masses, provided the temperature of the water were sufficiently low, and that it would be, if afforded by melting snow, or after having flowed over frozen ground. But there have been reports of large quantities of firm and solid ice having been found in this country at the bottom of deep and sluggish rivers, where there existed neither streams nor eddies to occasion the descent of frozen spicula from the surface of the water; and, if such ice have ever been found in such situations, it must be admitted, that it could not possibly have been conveyed there by the means above-mentioned.

I am, &c. &c.

T. A. KNIGHT.

XVIII. *On the action of detached leaves of plants. By T. A. Knight, Esq. F. R. S. In a letter addressed to the Right Hon. Sir Joseph Banks, Bart. G. C. B. P. R. S.*

Read June 13, 1816.

DEAR SIR,

SINCE I had last the honour to address a communication to you, with a request that you would lay it before the Royal Society, I have repeated great part of the experiments which formed the subjects of my former Letters, with such additions and variations, as might probably lead to the detection of any erroneous conclusions which I might have drawn: but I have not been able to detect any errors, nor to add any thing very important to my former observations. I have, however, been able to ascertain a few new facts, which I think too interesting to be lost.

I endeavoured, in my former communications, to adduce evidence, that the matter, which becomes vitally united to trees, previously passes through their leaves; and I shall now proceed to state some facts, which, I trust, will prove, that a fluid possessing the power which I have attributed to the true sap, actually descends through the leaf stalks.

A slender knife was passed through some leaf stalks of the vine, about two thirds of an inch distant from their junction to the branch; and, down to that point, the leaf stalks were divided longitudinally, and a transverse section, about half an

inch long, was made through the bark opposite the middle of the leaf stalk. A similar transverse section through the bark, was made somewhat less than an inch distant below; and these sections were united by two longitudinal sections through the bark, which extended from the extremities of the upper transverse sections to the extremities of the lower; by which means, pieces of bark, about half an inch broad, and nearly an inch long, were separated from the adjoining bark. These were then detached from the alburnum, and surrounded by two folds of paper coated with wax on each side; by which all connection and communication with the tree, except through the divided leaf stalks, were cut off. The insulated pieces of bark, nevertheless, continued to grow, and extended downwards, and laterally, and in thickness; and thin layers of alburnum were deposited.

Leaves of the potatoe, without any portion of bark being attached to them, were taken from the plants, just at the period when the tuberous roots began to be formed; and I conceived that these leaves, consistently with my former experiments and conclusions, must contain portions of the living organizable matter, which would subsequently have been found in their tuberous roots. The leaves were, therefore, planted in pots, and placed under glass, where, being regularly and properly supplied with water, they continued to live till winter, though without emitting fibrous roots; and I then expected to find some small tubers at their bases. In this expectation I was disappointed; but the result of the experiment was not less satisfactory, the bases of the leaf-stalks themselves having swollen into conic bodies of more than two inches in circumference, and being found to consist of matter apparently

similar to that which composes the tuberous roots of the plant. The enlarged parts of the leaf-stalks remained alive in the following spring; but whether they are capable of generating buds or not, I have not been able to ascertain.

Leaves of mint were planted in the same manner as those above-mentioned; which grew, and continued alive through the winter, and were still living in the end of the last month, having assumed the character of the thick fleshy leaves of evergreen trees. Upon examining the mould in the pots, I found it to contain very numerous roots, which must have derived their medullary, and their cortical, and alburnous substances from matter which had emanated and descended from the leaves.

I had frequently observed, in former experiments, that the destruction of the mature leaves of young plants not only suspended the growth of the roots, but also the growth of the immature leaves; whence I inferred, in a former communication, that the organizable matter, which composes the young leaves, has always undergone a previous preparation in other leaves of the plant, either of the same, or preceding season; and I was thence led to expect that, under favourable circumstances, the mature leaves might be made to nourish and promote the growth of immature leaves, without the aid of roots. Several shoots of the vine, each about a yard long, were detached from the trees, and laid over a succession of basins of water, into which each of the mature leaves was in part depressed; and thus circumstanced, the young leaves continued to grow, and the points of the shoots to elongate; and all were alive, and in perfect apparent health at the end of a month. The water necessary to preserve the young leaves

must in this case have been derived from the mature leaves; and I entertain no doubt, but that the organizable matter which occasioned their growth, was derived from the same source. Intersection of the bark between the mature and young leaves was not attended with any injurious consequences, and the sap must, therefore, have passed to the young leaves through the alburnum.

Consistently with the preceding circumstances, if the mature leaves be destroyed, or taken off, the fruit ceases to grow, or, if full grown, remains without richness or flavour; and the power of feeding fruits in winter and early spring seems to be confined to evergreen plants. The orange and lemon tree, the ivy and holly, afford familiar examples of this; and where a genus of plants consists of evergreen and deciduous species, as that of mespilus and viburnum, the evergreen species alone nourish their fruit in winter and early spring.

The probable passage of the sap from the mature to the young leaves and fruit, may, I think, be easily pointed out, though decisive proof of its course will probably never be adduced. Having often detached the bark from the alburnum of the stems of young oaks, just at the period when the midsummer shoots were beginning to elongate, I observed, as others have done, that a fluid exuded from those parts of the surface of the alburnum, which are called (most improperly) the medullary processes, and from correspondent points of the bark, which resemble the medullary processes in organization. This fluid has been proved, by its power of rapidly generating an organic substance, to be the true sap of the tree, part of which, I conceive, at this period, to be passing from the bark to join the ascending current in the albur-

num; which current feeds the young succulent shoots and growing leaves. Subjecting the alburnum to a slight degree of pressure at this period, I found that a considerable quantity of liquid, being apparently the true sap of the tree, issued out laterally through the medullary processes, as well as longitudinally through the cellular substance of the alburnum: but the tubes of it continued empty, and their position was marked by depressions of the surface of the extravasated fluid. I endeavoured to ascertain, what proportion of water a given quantity of the alburnum of such oak trees contained at this period; and I found that 1000 parts lost by drying only 871 parts: which is not more than the weight of the water that the cellular substance appears capable of containing, entirely independent of the tubes. That the tubes, nevertheless, are not always empty, but that they act at other periods of the year as reservoirs for the sap, I have given an opinion in a former communication; and I am now in possession of facts which prove them to perform this office, even in the heart wood, to a much greater extent than I had ever at any former period suspected; and which incline me to believe, that the durability of the heart wood, as well as of the alburnum of the oak, will be found to depend to a great extent upon the period in which the tree is felled: but I propose to make my observations upon these points the subject of a future communication.

I am, my dear Sir, &c.

T. A. KNIGHT.

XIX. *On the manufacture of the sulphate of magnesia at Monte della Guardia, near Genoa.* By H. Holland, *M. D. F. R. S.*

Read June 13, 1816.

The following account, which I have the honour to present to the Royal Society, of the manufacture of sulphate of magnesia at Monte della Guardia, near Genoa, has been drawn up chiefly from the note I made, when visiting this spot, in the spring of 1815. These notes have received additions from a paper, drawn up in 1803, by my friend Signore G. Mojon, lecturer on chemistry at the college of Genoa, who has bestowed much attention on the subject, and made several experiments on the composition and comparative purity of the manufactured salt.*

The Monte della Guardia, situated eight miles to the N. west of the city of Genoa, is one of the higher points in that part of the chain of the Ligurian Appenines, which borders immediately on the coast. The mountains in this portion of the chain are chiefly of primitive slate, with some subordinate formations of marble and serpentine, a considerable extent of transition and secondary limestone, chiefly to the eastward of Genoa, and a few more partial deposits of sandstone and coal.

The summit of Monte della Guardia is somewhat more than 2000 feet above the level of the sea, from which it is five

* This paper was published by the Society of Medical Emulation in Genoa.

miles distant. The ascent towards the mountain from Sestri, the nearest point on the coast, is uniform, though rugged; following the course of the rapid torrent, called the Panigaro. The ravine of this torrent is interesting, inasmuch as it forms the division between a high ridge of serpentine, which occurs on its eastern side; and an insulated formation of magnesian limestone, which composes the hill of Monte del Gazzo, to the north of Sestri, and other adjacent heights on the western side of the stream. This magnesian limestone, which in Monte del Gazzo reaches a height of more than 1200 feet, forms in various places naked cliffs, distinctly showing the stratification of the rock; which stratification, as well as the colour of the limestone, is very remarkably contrasted with the character of the serpentine rocks, on the opposite side of the ravine.

The latter, together with some argillaceous and chlorite slate of the same formation, compose for several miles, the ridge of hills between the torrent of Panigaro, and the broader valley of the Polcevera to the east; stretching upwards in a direct line from the sea towards the summit of Monte della Guardia. The primitive schistus is doubtless the base of this formation; the serpentine reposes upon it in vast unconformable masses; exhibiting a surface of great ruggedness, and almost entirely bare of vegetation; with the dark green colour, and usual resinous lustre of this mineral. I observed much steatite, talc, and asbestus in it; also many small veins of pyrites. The asbestus of this place affords numerous fine specimens to the cabinets of Genoa.

Monte della Guardia may be considered the summit of this ridge. It exposes to the south a somewhat concave outline;

the front of which is deeply worn out, so as to show the stratification of the primitive slate, mixed with chlorite slate, with the other magnesian minerals just noticed, and with numerous veins, or layers of copper and iron pyrites. The specimens of this pyrites have a steel grey, or greenish yellow colour, according to the respective predominance of iron or copper. The whole substance of the pyrites has a schistose structure, corresponding with that of the rocks in which it is situated. It is so intimately mixed with the same magnesian minerals, as to be for the most part extremely unctuous to the touch. It is in general easily sectile. The specific gravity varies from 3.6 to 4.6.

It is on this side of the Monte della Guardia, and from the materials just described, together with the magnesian limestone of Monte del Gazzo, that the manufacture of sulphate of magnesia has been established; it is at present carried forward on a small scale, but is evidently capable of extension and improvement. The original object of attention in this place was the working of the copper and iron pyrites; but the observation of the crystals of sulphate of magnesia, formed during the processes applied to these ores, changed in great degree the character of the manufacture; and the green vitriol and copper obtained here are now become secondary objects to the proprietors. The process employed for the formation and separation of the sulphate of magnesia is briefly as follows: the pyrites is worked out of the mountain by tunnels, which the steep natural section of its front, and the general direction of the veins, allow to be carried nearly on an horizontal level. Some of these galleries, which I saw, were more than 200 feet in length, and varying from 10 to 15 feet

in width. Others, which are now filled with water, or destroyed by the crumbling down of the rock, were described to me as of still greater length. The ore, thus obtained, is broken down into small pieces, and roasted in an open pit, or kiln, about 20 feet in depth; with the alternation of wood, so disposed as most effectually to aid the combustion. This process of roasting continues generally for ten days. The kiln being sufficiently cooled, the ore is removed thence, and disposed in large heaps, underneath a shed in the vicinity. Here it remains for several months fully exposed to the air, and occasionally moistened with water thrown upon it, to aid the chemical changes by which the salt is found. An efflorescence of sulphate of magnesia soon commences, and gradually proceeds, so as to cover with minute crystals the surface of the ore, which, during this time, crumbles down into very small fragments.

This is the usual commencement of the process, but there are varieties of the ore, more easily decomposable, which require less roasting, or may even be submitted to the after processes, without any application of heat. Other varieties, on the contrary, require more roasting, to prepare them for the proper efflorescence of the salt, which efflorescence itself varies in rapidity, according to the particular description of the ore.

The materials, brought into this strata, are then lixiviated; and the liquor containing in solution the sulphates of magnesia, of copper, and of iron, is filtered; in passing it through layers of sand and straw, disposed in large wooden vessels. These processes repeated as often as may be necessary, the next object is the separation from the liquor, of the metallic

sulphates. When the sulphate of copper is perceived to be abundant, the metal is precipitated by refuse of iron, introduced into the liquor. The sulphate of iron, and any remaining sulphate of copper, are decomposed by the addition of lime, in the state of milk of lime, which causes a precipitate of sulphate of lime, together with the metallic oxides.

The lime employed for this decomposition, is obtained from the magnesian limestone, already described as belonging to this district. That of Monte del Gazzo is preferred, both as nearer to the spot, and possibly from its containing a larger proportion of magnesia, than the other magnesian limestones of this coast, about 16 per cent., as appears from analysis made of it. It appears probable, that a certain portion of sulphate of magnesia is obtained from the use of this limestone, in decomposing the metallic salts, and the process of manufacture thereby rendered somewhat more productive. The quantity of lime employed may equal about $\frac{1}{100}$ part of the weight of the pyritic ore.

The metals being thus separated from the saline liquor, it is filtered anew, and then evaporated to a certain point in a large copper boiler. It is subsequently let off into small glazed earthen vessels where the crystallization of the salt takes place. M. MOJON mentions the circumstance of its being needful frequently to disturb the crystallization, to satisfy the prejudice of the purchasers, who consider it essential that the salt should be delivered to them in the form of small spicular crystals.

The produce of sulphate of magnesia varies of course according to the quality of the ore. In general, perhaps, it may be stated at $\frac{1}{10}$ of the weight of the material employed. The

ore remaining after the first filtration, is usually roasted and lixiviated a second time, to obtain the portion of salt which it is still capable of affording.

It is almost unnecessary to make any remarks on the theory of the process just described. The sulphuric acid, formed by the action of heat, air, and water upon the pyritic ore, combines with the metals, and with the magnesian earth, which the ore contains. I am not aware of any experiments that have been made, to ascertain the proportion of salt which may be derived from the use of the magnesian limestone. Probably, however, it is extremely small; and it would seem very doubtful, from the nature of the process, whether any improvement to the manufacture can be expected from this source. The effect of employing more of the limestone than is strictly necessary, would be, that the lime, rather than the magnesia, would unite with the sulphuric acid; and if the addition of the mixed earth were in excess, the lime alone would have effect in the decomposition; and sulphate of lime, with the metallic oxides, would be the new products obtained.

M. Mojon has proposed a variation in the manufacture at Monte della Guardia; depending on the different tendency of the sulphate of magnesia, and of the metallic sulphates towards crystallization. The principal object of the variation would be to save the sulphate of iron, now lost by decomposition; and to increase the value of the produce of copper by obtaining it in the state of sulphate. This change in the process has not hitherto been adopted; and probably might be found to be attended with several difficulties in practice.

The buildings connected with the establishment at Monte

della Guardia, are on a small scale; their situation is a singular one, at the height of 1600 feet above the level of the sea. The present conductor, and, as I believe, the chief proprietor of the establishment, is Signore ANSALDO of Sestri, a man of intelligence and activity.

The produce of the sulphate of magnesia, as far as I could ascertain from enquiries on the spot, does not at present exceed 1½ cwt. per week. It might doubtless be increased to a much larger amount, from the abundance of the materials of manufacture. The colour and general appearance of the salt are good, and M. MOJON has found it by analysis to be a very pure sulphate of magnesia. It is used pretty extensively in the medical practice of Italy, under the name of the Sal Inglese; and might, in the extension of the manufacture, become an article of very considerable profit to the proprietors of the establishment.

Before concluding this paper, I may add, that I have been informed of an analysis by VAUQUELIN of the pyritic ore of the Monte della Guardia; but, from the want of sufficient references, I have hitherto been unable to ascertain the results of this examination.

XX. *On the formation of fat in the intestine of the tadpole, and on the use of the yelk in the formation of the embryo in the egg.* *By Sir* Everard Home, *Bart. V. P. R. S.*

Read May 23, 1816.

THE tadpoles in England are so small, that no person has attempted to make an accurate investigation of the internal changes of structure that take place in them, between the time of the animal leaving the egg and becoming a frog.

The rana paradoxa of Surinam, in its tadpole state, is larger than in any other species of frog yet known, and so closely resembles fishes, that in that country it is sold as such in the market, for the use of the table, under the name of Jackie; and, as the frog produced from it, is in the first instance as large as a common frog in this country, it is highly probable, that it becomes an animal of considerable size. Mr. IRELAND, a surgeon in the army, who resided several years at Surinam, and watched the tadpole through many of its changes, has brought home specimens in different stages of its metamorphosis. These he has kindly submitted to my examination, and has in the most liberal manner, deposited the specimens in the Museum of the Royal College of Surgeons in London, of which he is a member.

As Mr. IRELAND had no opportunity of examining the tadpole, before the hind legs begin to make their appearance through the skin; with a view to render the series complete, I have since examined the progress of the changes, from the

ovum in the spawn of the English frog.* The jelly of which the ova are composed, I have upon a former occasion given an account of to the Society. The ova themselves differ from those of snakes and lacertæ in general, in having no yelks. When the tadpole is once formed, it appears to feed upon the jelly, which although not absolutely albumen, is a near approach to it.

In this stage, each ovum is pressed into the form of an hexagonal prism with flattened ends, so as to form the whole into one compact mass.

The tadpole, after it leaves the ovum, has on each side ten filaments projecting from the neck, for the purpose of aeration of the blood; such filaments must be considered as temporary gills.

The lacerta of this country, called the newt or eft, in its larva state, has the same projecting filaments, which drop off when the gills are formed; they are more complex in their structure, and only three in number, on each side. This circumstance shows, that the larva of an eft is a species of tadpole, and that the eft itself does not belong to the tribe of lizards, but is a nearer approach to that of frogs. In the tadpole, as soon as the abdomen begins to enlarge, these external filaments disappear. Twenty-four similar filaments are met with in the fœtus of the shark while contained in the egg, which drop off before the fœtus escapes from the shell.

The spawn of the English frog was collected on the 1st of April, 1816. On the 15th, the tadpole left the egg, but the filaments or external gills were not visible, only a deep notch

* The tadpoles having become frogs since this Paper was read, I have been enabled to complete the series.

on each side nearly separating the head from the body. On the 23d, the ten filaments on each side were distinct; on the 27th they disappeared. In June, the external orifice on the left side, for the water to pass off from the gills, was very distinct, but none was seen on the right. On July the 8th, the hind legs began to appear, but the toes were not separated. On the 14th of July, the hind legs were seen externally completely formed, and on opening the skin of the chest, the fore legs were equally so; but there was no external projection by which this could be known. The lungs were completely formed. On removing the intestine, there was no fat deposited on the loins. On the 16th, the contents of the intestine were voided in considerable quantity. On the 18th, the elbows of the fore legs projected under the external skin, and so much of the contents of the intestine had been voided as to give a taper form to the lower part of the body. On the 19th, the fore legs were completely disengaged and appeared externally; the mouth had become wide like that of a frog. The tail had a notch at that part where it afterwards separates; the intestine was reduced in diameter, and to the length of that of a frog; an appearance of oil was seen on the loins. On the 23d, the tail had dropped off, leaving the projecting root. The animal had left the water and remained among the grass. Behind the intestines upon the loins were several small membranous appendages in an empty state.

On the 28th, the root of the tail had wholly disappeared; the appendages had become more opaque.

The ova of the frog appear to be hatched at very different periods, since some of the tadpoles become complete frogs, before others have their hind legs protruded through the skin.

Upon examining the tadpole of the rana paradoxa, just

when the hind feet appear externally, I found the mouth very small, and nearly round, the teeth cuticular, the upper ones overlapping the under, the œsophagus, stomach, and intestine, forming one uniformly continued canal, which passed down to the lower part of the abdomen; it was bent upon itself, passed up again, and then made a great number of coils in a circular form; its coats were very firm, its capacity very small. There were three gills completely enclosed on each side, and a little way below the eye on the left side, a small round orifice, for the water, by which the gills are supplied, to pass out; but none on the right. When the tadpole is arrived at its full growth, and the hind legs are completely formed, which takes place, according to Mr. IRELAND's observations, in 14 days after their first appearance, the cavity of the abdomen had become exceedingly enlarged, the intestine very capacious, its coats almost as thin as cobweb: it was completely distended, through its whole extent, with a soft substance, which when burnt had the smell of hay. Behind the intestine, all along the posterior part of the abdomen, a large quantity of fat was met with of a yellow colour, enclosed in long, thin, transparent membranous bags; no part of this fat was met with in the prior stages of the tadpole's growth. The lungs were completely formed.

When the mouth of the tadpole has been changed into that of the frog, and the fore legs completely protruded, but the tail remaining entire, which happens 21 days after the last mentioned change, the large coils of intestine were found contracted into a canal one fourth of its original length; the coats had become as firm as those of an artery, the external surface was corrugated and the canal empty. The stomach had become a distinct cavity, and there was a contraction, where it

terminates in the intestine. All these parts were embedded in fat, which filled every part of the abdomen, not occupied by the liver, which had acquired a large size. The lungs were filled with air, and the gills had entirely disappeared.

When the tail has dropped off, leaving the projecting root, which takes place in seven days more, the only internal change met with, was, that no fat whatever was found in the cavity of the abdomen.

The great length of the intestine which has been described, has nothing analogous to it in the caterpillar, and is probably confined to the frog tribe.

The egg of a frog bears no proportion in size, to those of the other animals of the same class, and differs from them in having no yelk, therefore, although it contains sufficient materials for the formation of the tadpole, something is still wanting, before it can be metamorphosed into a frog; and in the tadpole state, a store of fat is laid up, beyond what is required for its own immediate support and future growth, to furnish the necessary means of supplying the different structures in the frog, not already existing in the tadpole; and this fat appears to be formed in the intestine.

The length of intestine in the tadpole, when its relative proportion to the size of the animal is considered, exceeds every thing of the kind that is met with in other animals.

In the tadpole of the Surinam frog, the intestine after it has acquired its full size, does not remain of this enormous length, beyond the period of its metamorphosis into the frog taking place; and what is deserving of particular attention, the fat is deposited, when the intestine has acquired its full size, and no sooner is the intestine reduced in length, than not only no

more fat is deposited, but all that was previously formed is found to have been consumed, in producing the metamorphosis into the frog: which leads me to conclude, that such a deposit of fat is necessary to the metamorphosis of a tadpole into a frog, and that such unusual length of intestine, is required to admit of so large a quantity of fat being formed in so short a time, and, therefore, that the intestine is the laboratory in which the fat is formed.

To ascertain, whether the necessity of such a supply of fat is occasioned by the soft parts of the tadpole not being convertible into bones, and other parts of the frog, which did not exist in the tadpole, or, simply, from a deficiency of materials, I have had the assistance of my friend and fellow labourer in animal chemistry, Mr. HATCHETT, who some years ago ascertained, that the yelk of an egg is essentially composed of concrete oil, combined with a small proportion of albumen, and he has made out the following important facts. "That the spawn of the frog, has no yelk, and contains no oil whatever; he also corroborates Mr. BRANDE's statement, that it consists of a substance intermediate between albumen and gelatine, inclining principally to the former. That the ova of the shell snail, both of those that have a shell, and those that have only a strong membranous covering, have no yelks, and consist of albumen, since they coagulate in proof spirit of wine, and, when so coagulated, and examined some time afterwards, appear not to contain any oil. That the ova of the lobster have no yelk, and contain no oil.

But he remarks, that the spawn of the lobster, when recent, is filled with albumen mingled with a substance of a dark olive colour, and whilst the former as usual is coagulated by heat,

the latter becomes of a vivid red; this, Mr. HATCHETT observes, is the colouring matter of the shell, which three or four years ago, Mr. BRANDE found to become red by the application of acids without heat; for dilute sulphuric, nitric, muriatic, and the strong acetic acids immediately produce the same effect on the colouring matter of the spawn, but this is not the case when a weak acid, such as common distilled vinegar, is employed.

When this bright red colour has been produced by the above acids, it appears to be permanent, excepting when nitric acid has been used, for then the red colour changes to yellow, which by the affusion of ammonia becomes orange colour, as is usual with animal substances so treated.

Dilute nitric acid in which it had been digested, afforded slight traces of a phosphate, which was not phosphate of lime.

As the red colour is produced by acids as well as by heat, there was some reason to expect that it would have been destroyed, or at least that its intensity would have been diminished by a great excess of the alkalies, but not the smallest effect was produced by any of these, and indeed so far from it, that the recent spawn when put into a solution of caustic potash, became in a few seconds changed to as bright a red as when the mineral acids had been employed.

The red colour is also produced by the effects of air, light, and the evaporation of moisture, for paper or linen which have been stained with the olive coloured substance, become red in the course of a few minutes, so that in this respect it somewhat resembles the secretion obtained from the buccinum lapillus, or purple whelk. The purple colour of this last does not, however, suffer any change, whilst the colouring

matter of the lobster in the course of some days becomes of an ochraceous colour. In this state it seems to be permanent, for it was retained by linen which had been marked with it, after repeated boiling in water and washing with soap.

From these cursory experiments, Mr. HATCHETT observes, that this animal colouring substance is apparently of a peculiar nature, and that it is the same in the common cray fish and the prawn, as well as in the sea cray fish and the crab, but in the two last, it has already assumed its red colour.

That the ova of the salmon and pike have no yelk, and consist principally of albumen, as they coagulate by heat, but contain also a small portion of oil, which perhaps is a substitute for yelk.

That the ova of the cartilaginous fishes, as well as those of the lizard and snake, have a regularly formed yelk, like that of the hen, composed of the same ingredients; but in both the viviparous and oviparous sharks there is no perfectly formed albumen, but in its place a gelatinous substance, which Mr. BRANDE ascertained to be intermediate between gelatin and albumen, similar to what is met with in the spawn of the frog.

In addition to that which has been above stated, Mr. HATCHETT has communicated to me the following observations.

"The yelk of the eggs of birds is principally and essentially composed of a butyraceous oil, combined with a small proportion of albumen, the average of which in the yelk of the common fowl amounts to about one fifth. Yelk, when triturated and diluted with water, forms (as is well known) an emulsion, and yelk may be regarded as an emulsion in a high state of concentration.

In milk, the caseous part, or curd, corresponds to the albuminous part of yelk, as the butter in milk does to the other part or oil of the yelk. The principal difference, therefore, between milk and yelk is, that the former is in a dilute, and the latter in a concentrated state. Hence, Mr. HATCHETT observes, it appears that many of the oviparous animals during the period of incubation, are nourished by a pabulum similar in quality to that by which young viviparous animals are supported, whilst the great degree of concentration of this pabulum in the first case is essentially necessary, in order that the quantity of nutritious matter which is required during incubation, and which is included with the animal within the egg, should be condensed into the smallest possible bulk.

Young viviparous animals are at first incapable of supporting themselves by those substances which are afterwards to become their food, and they are therefore nourished for a certain period of time by the milk of their mothers; but young oviparous animals, such as the chicken, partridge, and birds in general, come forth from the shells complete in their bodily faculties, and immediately partake of the food to which the parent birds are accustomed, so that it seems they are prepared for this, and are nourished during incubation, by a substance similar in its nature to that by which young viviparous animals are supported, or suckled, during a certain time after their birth, and that the process corresponding to that of suckling, is, with regard to birds, performed and completed in the course of incubation.

The experiments which Mr. HATCHETT has made upon the ova of these different tribes of animals, lead to the conclusion, that in all ova, the embryos of which have bones, there is a

certain portion of oil, and in those ova whose embryos consist entirely of soft parts, there is none.

This conclusion is much strengthened by the peculiarity, which it has been my object in this paper to point out, of the tadpole laying up a magazine of fat before the metamorphosis into a frog takes place; it is, therefore, rendered probable, that a certain portion of oil is necessary for the formation of bone, and that the proportion in different ova corresponds with the greater or less degree of hardness of the bones of the fœtus.

XXI. *On the structure of the crystalline lens in fishes and quadrupeds, as ascertained by its action on polarised light.* By David Brewster, *LL.D. F. R. S. Lond. and Edin. In a Letter addressed to the Right Hon. Sir Joseph Banks, Bart. G. C. B. P. R. S.*

Read June 20, 1816.

DEAR SIR,

THERE is, perhaps, no subject in natural history which has excited so much attention, as the structure and functions of the eyes of animals; and there is certainly none which has so ill repaid the anxiety and labour with which it has been investigated. The physiologist was naturally led to study the mechanism of an organ through which man receives the noblest of all his enjoyments; and the natural philosopher, considering it as the work of infinite intelligence, ardently anticipated the improvement of optical instruments from the imitation of this perfect model. It is discouraging however, to estimate the real amount of the labours of the one, and to perceive how little advantage has been derived from them by the other. The most prominent functions of the eye are still very imperfectly understood, and the improvement of the telescope has been retarded, rather than advanced, by the pursuit of a false analogy. I have, therefore, some satisfaction in being able to throw additional light upon a subject of such difficult investigation, and so generally interesting both from its optical and physiological relations.

Having found that the doubly refracting structure could be communicated to glass and other bodies, by giving them a variable density either through the agency of heat or mechanical pressure, I was led to conclude, that the same structure would be found in the crystalline lens of fishes and other animals, which was known, by direct experiment, to increase in density towards the centre. I had formerly examined the action of the crystalline upon polarised light, without obtaining any new result; but I now placed such reliance on the truth of the general principle, that I resumed the subject with the utmost confidence of success.

Upon exposing to a polarised ray the crystalline lens of a large cod, included in its capsule, I could not perceive, as happened in my early experiments, any very distinct indications of a peculiar action. I plunged it, however, in Canada balsam contained in a hollow parallelopiped of glass, and was surprised to observe a regular optical figure varying its shape during the revolution of the crystalline.

I now turned the spherical crystalline, till the diameter which corresponded to the axis of the eye, or the line joining the poles to which the fibres converge, was parallel to the polarised ray, and I observed the appearance shown in Fig. 1. (Pl. XII.) consisting of twelve luminous sectors, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, separated from each other by a black cross, and by two dark concentric circles. The interior sectors, 1, 2, 7, 8, were small and exhibited a white tint of the first order, increasing in brilliancy towards the centre. The middle sectors 3, 4, 9, 10, which are very large, are separated from the interior ones, by a broad dark circle, and display a white tint of the same intensity. The outer sectors

5, 6, 11, 12, are extremely faint, and are seen with considerable difficulty in this position of the lens.

If the crystalline is now turned round, so that its axis, which corresponds to the axis of vision, may always be parallel to the polarised ray, the same appearances will be seen without the slightest variation. But if this axis is inclined to the polarised ray in the direction 1, 2, the sectors 1, 2, will diminish, and 7, 8, will increase in size, and an additional white spot will appear at the centre as in Fig. 2, till by increasing the inclination, the sectors 1, 2, and the white spot will completely disappear, leaving the sectors 7, 8, much enlarged and of a bluish white tint. If the inclination is in the direction 7, 8, the sectors 1, 2, will increase, and 7, 8, will diminish in the same manner.

By transmitting the polarised light through other two faces of the glass parallelopiped, so as to traverse the crystalline in a line perpendicular to its axis, the optical figure presented new appearances. When the axis of the lens was either parallel or perpendicular to the plane of primitive polarisation, which happened four times in the course of a revolution, it exhibited the form shown in Fig. 3. The tints 1, 2, 7, 8, were now reduced to a pale blue of the first order, and the black cross was very ill defined at the centre. The middle sectors 3, 4, 9, 10, were a little reduced in size, while the exterior ones 5, 6, 11, 12, had experienced a very sensible augmentation. At intermediate positions of the crystalline, when its axis was inclined 45°, 135°, 225°, and 315° to the plane of primitive polarisation, the optical figure assumed the appearance shown in Fig. 4, where the two sectors 7, 8, are greatly enlarged, and the other two 1, 2, have wholly, or almost wholly, disappeared.

I now removed the capsule of the lens, so as to let out the semifluid matter which it enclosed, and having rubbed off the very soft exterior coat, I immersed the diminished sphere in Canada balsam, but I could never observe the exterior sectors, 5, 6, 11, 12, all the other appearances being exhibited as before. I next removed the middle coats of the crystalline, and replacing the nucleus, which was now reduced to one-eighth of an inch in diameter, in the glass parallelopiped, I observed the central sectors 1, 2, 7, 8, without any of the middle or exterior ones. By pressing the nucleus between two plates of glass, or by allowing it to indurate gradually, the depolarised tints ascended in the scale of colours, as in the case of animal jellies.

If we now take a plate of sulphate of lime which polarises a blue of the second order, and combine it with the crystalline lens, so that its axis may be parallel to the line 6, 10, 2, 1, 9, 5, the white tints 9, 10, will ascend to a green of the second order, while those at 1, 2, 5, 6, will descend to an orange red of the first order. In like manner, if the axis of the plate of sulphate of lime is parallel to the line 11, 3, 7, 8, 4, 12, the tints 3, 4, will become green, and 7, 8, 11, 12, an orange red.

Hence, it follows, *that the nucleus* 1, 2, 7, 8, *and the exterior coat*, 5, 6, 11, 12, *have the same structure as one class of doubly refracting crystals*, while the middle coats 3, 4, 9, 10, have the structure of the other class.

In order to compare these different structures with those of glass crystallized by heat and by pressure,* I took a polished sphere of crown glass, and, having brought it to a red heat, I cooled it by rolling it quickly in every direction over a smooth surface. When it was immersed in Canada balsam,

* See Phil. Trans. 1816, p. 46, 156.

and exposed to polarised light, it had the appearance shown in Fig. 5, in whatever position it was held, the highest tint being an orange yellow of the first order. By examining these sectors with sulphate of lime, I found that the glass had the same structure as the middle coats of the crystalline. In like manner, it appeared that the sectors, exhibited by pressing a convex lens upon a flat piece of glass, were produced by a structure the same as that of the central nucleus of the crystalline. The structure of the crystalline lens, in short, is similar to that of a plate of glass that gives the unusual fringes* bent into a circular shape. Hence it follows, *that the central nucleus and the external coat are in a state of dilatation, while the intermediate coats are in a state of contraction, and that these opposite states are not dependent upon each other as in crystallized glass.*

The phenomena which have now been described are visible also in the crystalline of the *haddock*. They appear likewise in that of *sheep* and *oxen*, but there is here only one series of luminous sectors corresponding with the intermediate set in the crystalline of fishes. The human crystalline will no doubt display similar properties, but in an inferior degree.

The *cornea* both of fishes and quadrupeds, and also the human cornea, have an analogous structure; in which the optical axes of all the particles are directed to its centre. Its structure is the same as that of the internal nucleus, and it produces an effect upon polarised light similar to what is shown in Fig. 6.

The *sclerotic coat* of fishes has the property of depolarising light in separate spots like the diamond, or a mass of crushed

† See Phil. Trans. 1816, p. 65, 66.

isinglass; but it derives this property from a bluish white membrane which covers the outside of it, for when this is removed, it loses the doubly refracting structure. If the sclerotic coat is boiled, it is capable of receiving the structure of doubly refracting crystals by mechanical compression and dilatation. In its natural state, it possesses the same property, but in an inferior degree. The cornea is also capable of having its doubly refracting force increased by compression or dilatation.

From these experiments the following conclusions may be deduced.

I. All the parts of the crystalline lens of fishes corresponding to the two dark concentric circles, exercise no action upon polarised light. The outward spherical shell which acts upon light like one class of doubly refracting crystals, and also the solid nucleus which exercises a similar action, are in a state of mechanical dilatation, while the middle spherical shell which acts upon light like the other class of crystals, is in a state of mechanical contraction.*

II. The structure of the crystalline lens in fishes is not symmetrical, as has hitherto been supposed, consisting merely of a number of coats of different densities; but it has a distinct relation to that diameter of the sphere which is the axis of vision.

III. The variations of density which produce the doubly refracting structure, are not related to the centre of the crys-

* When the crystalline lens is examined by common light, there is an obvious appearance of a rapid change of density at the line which separates the middle and the exterior sectors. This is probably the boundary of the fluid coat adjacent to the capsule.

Fig. 1.

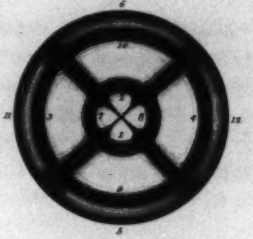


Fig. 3.

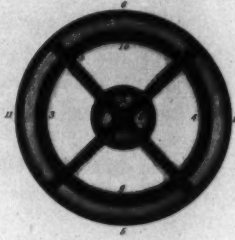


Fig. 4.

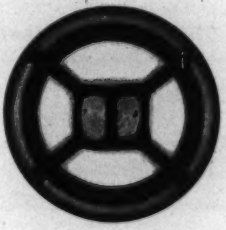


Fig. 2.

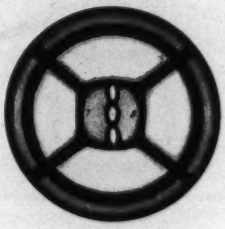
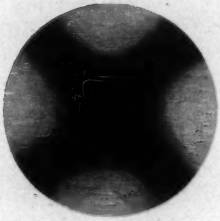


Fig. 5.

Fig. 6.



talline, but to the diameter which forms the axis of vision. For if the variation of density were related to the centre, the sphere would have a symmetrical structure, and like the glass ball already mentioned, would exhibit the same figure in every position.

IV. It is highly probable, that this peculiar structure of the crystalline is necessary for correcting the spherical aberration.

I have the honour to be, &c.

D. BREWSTER.

To the Right Hon. Sir Joseph Banks, Bart. G. C. B. P. R. S.

Edinburgh, May 20, 1816.

XXII. *Some farther account of the fossil remains of an animal, of which a description was given to the Society in* 1814. *By Sir* Everard Home, *Bart. V. P. R. S.*

Read June 13, 1816.

For the materials of the former Paper I was indebted to Mr. Bullock, who has in his possession the skull, a great number of the vertebræ, many mutilated ribs, and other bones of this animal, in a fossil state.

These now brought forward are in the collections of the Rev. Mr. Buckland, of Corpus Christi College, Oxford, and of Mr. Johnson of Bristol, who have very kindly allowed me to make use of them upon this occasion. Mr. Johnson has been a collector of specimens of fossil remains for 25 years; during several summers, he devoted five or six weeks at a time to a close inspection of the cliffs and beach at Lyme. In the summer of 1814, with the assistance of a friend, at great personal risk, he dug out of the cliff, the bones of the pectoral fin; the single bone, he states, was immediately connected with the scapula, and was imbedded in marle; a representation is annexed.

From these valuable specimens I am enabled, in a great measure, to complete the account of the skeleton of this very extraordinary animal, and, what is of infinitely more consequence, to determine the class to which it belongs.

The structure of the vertebræ explained in the former paper, made it evident, that the animal in its mode of progressive motion resembled fishes; it could not, however, be determined that it was in all respects a fish, till the articulations of the ribs with the vertebræ, and the bones of which the pectoral fin is composed, had been examined.

In all animals that breathe by means of lungs, each rib, to admit of its being raised and depressed, is articulated both to the body and the transverse process of the vertebræ; but in fishes, the ribs requiring no such motion, are only connected to the bodies of the vertebræ laterally, so as not to interfere with their extensive motion on one another. In this animal, the ribs are placed in this respect like those of fishes; they are uncommonly large, and the chætodon from Sumatra, the skeleton of which is described by Mr. BELL in the first part of the 83d vol. of the Philosophical Transactions, is the only fish I know of in which the ribs bear the same proportion to its size. The form of the scapula, as well as of the bones of the pectoral fin, is entirely different from those of the whale, but bears a resemblance to that of the same parts in the shark, so that it is only necessary to compare them together as represented in the annexed plates, to recognise their similarity.

The other circumstances that confirm this skeleton being that of a fish, are the bones in a growing state having no epiphyses, as will appear from the first bone of the pectoral fin, which is represented of its natural size, having none, although when compared with the single vertebra, also represented of its natural size, the fin must have belonged to a growing animal; the ribs having been grooved longitudinally by pressure, showing the softness and toughness of their

texture, the fibrous appearance of the scapula, arising from a mode of growth only met with in the bones of fishes.

The drawings annexed to this and to the former Paper, represent the principal bones composing the skeleton of this very extraordinary animal, and they correspond sufficiently with those of fishes, to remove all doubt of its having been a fish, but different from any fishes now in existence; for although the pectoral fins have a certain resemblance to those of the shark, there is none between many of the other parts, particularly the long projecting snout and the caudal fin.

In truth, on a due consideration of this skeleton, and of that represented in the 13th vol. of the Am. Phil. Trans. we cannot but be inclined to believe, that among the animals destroyed by the catastrophes of remote antiquity, there had been some at least that differ as widely in their structure from any which now exist, as to render it impossible to arrange their fossil remains with any known class of animals.

EXPLANATION OF THE PLATES.

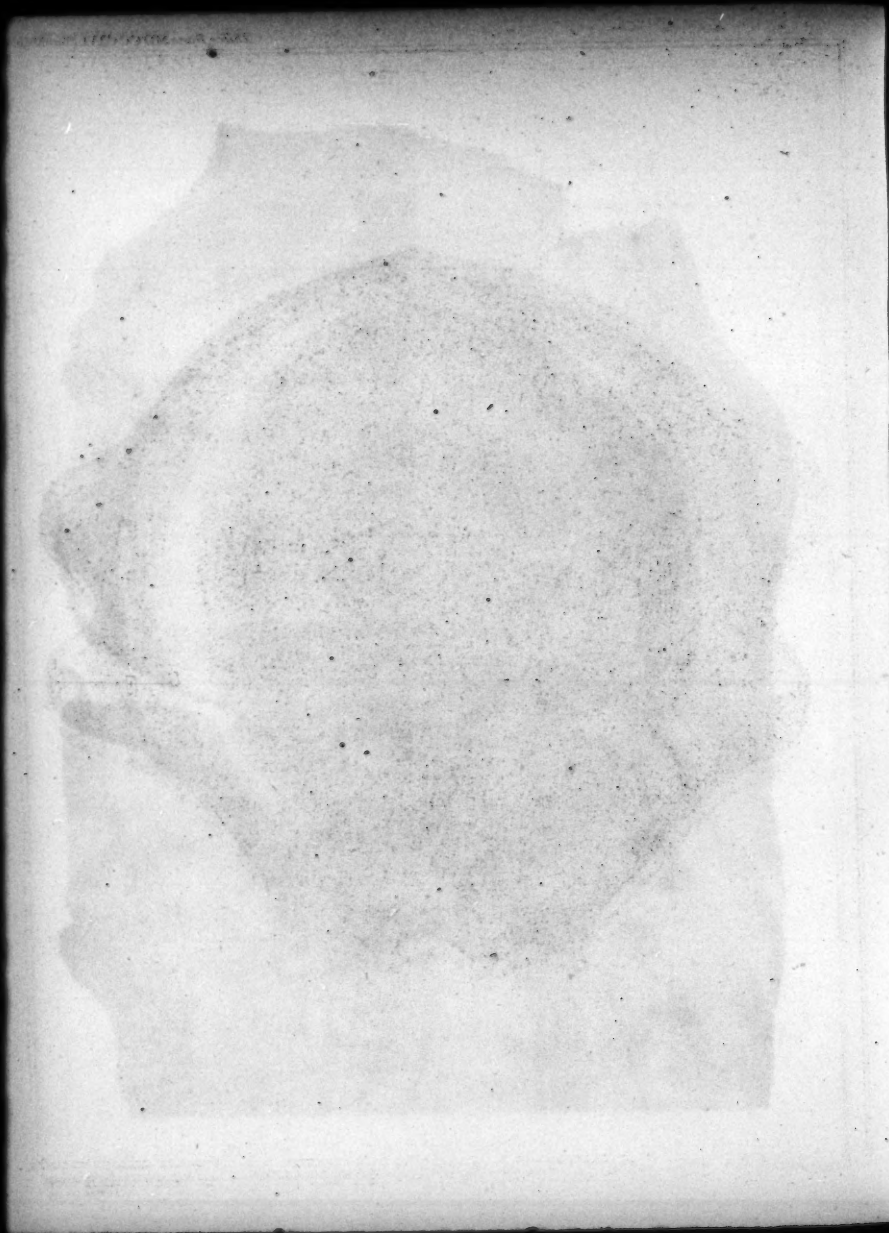
The drawings were taken from specimens of bones in very different stages of growth, but probably belonging to the same species of animal.

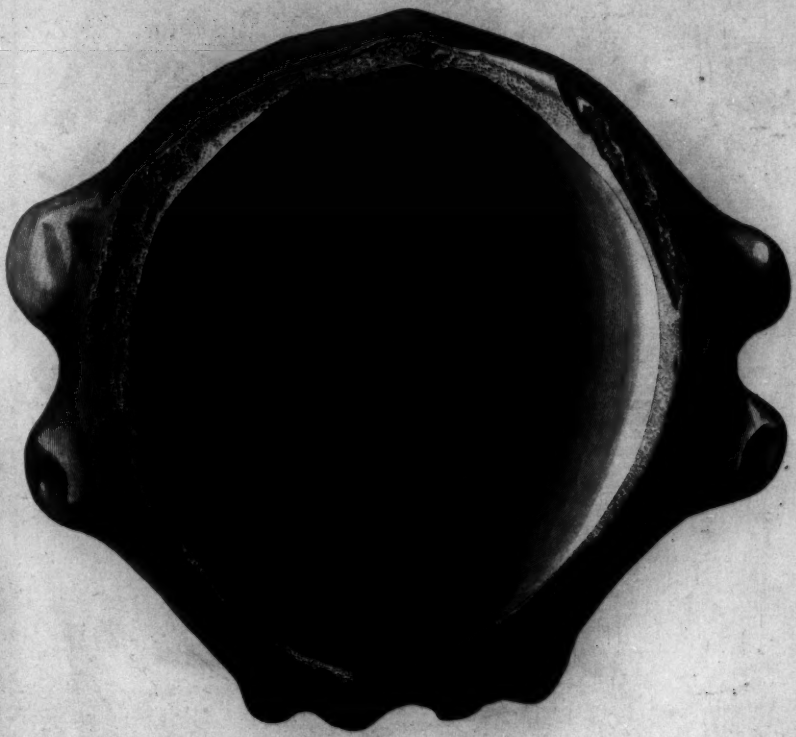
Plate XIII.

Shows the manner in which the bones of the fins correspond with the impressions in the matrix destined to receive them. The parts of the animal found ... Rees, a surgeon of the Rev. Mr. Bucklands.

Plate XIV.

A single vertebra of the natural size, a shows that the





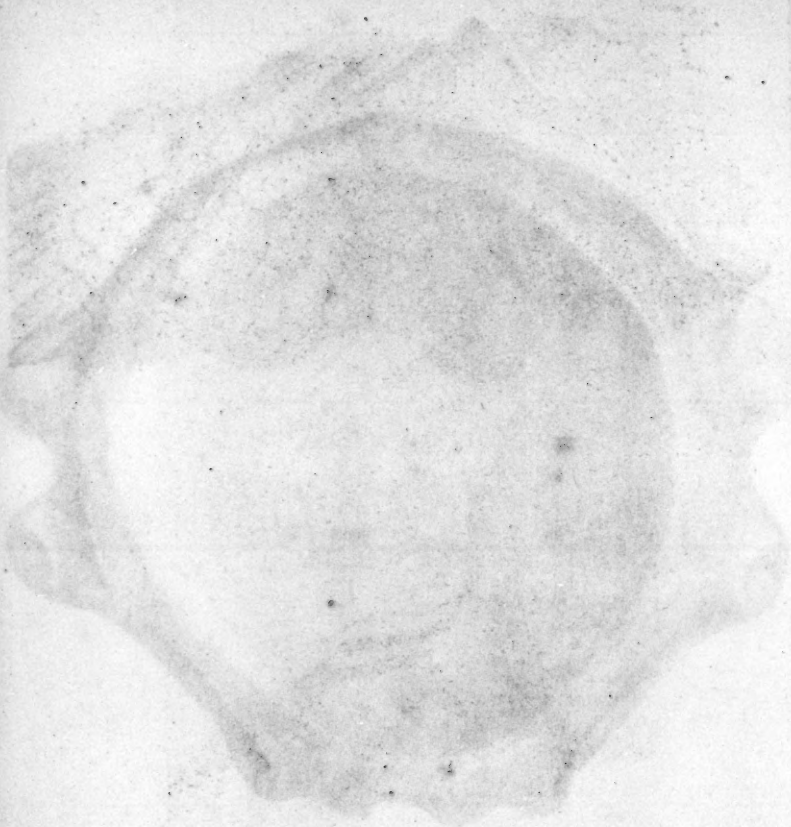
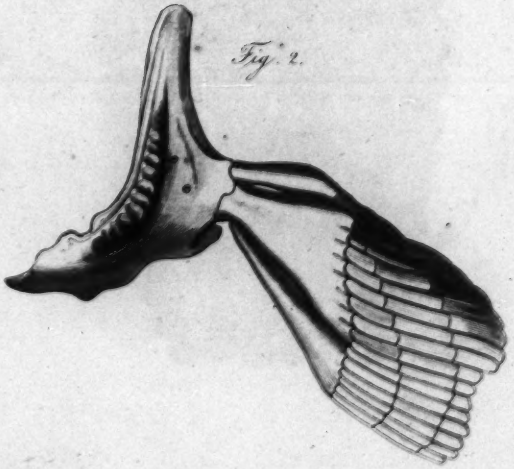
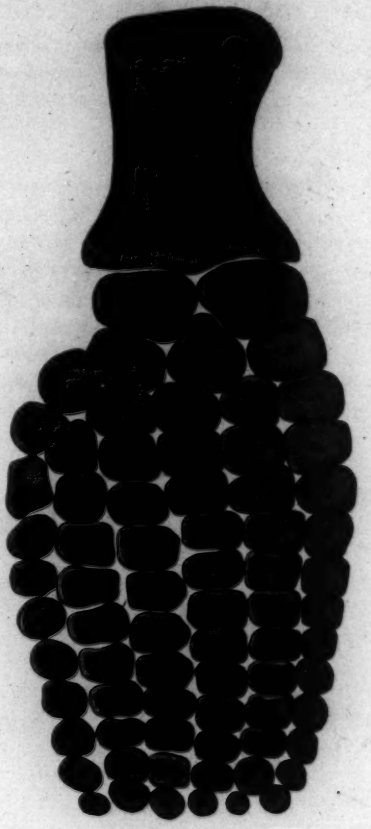
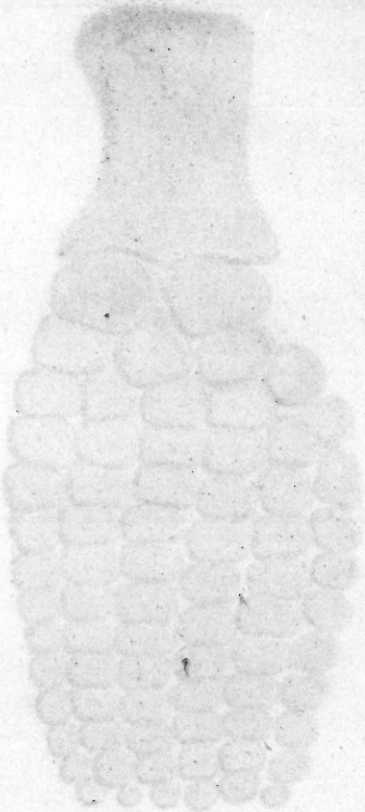


Fig. 1.

Fig. 2.







animal grows to a large size, and points out the particular parts of the circumference to which the rib is attached. From a specimen of the Rev. Mr. BUCKLAND.

PLATE XV.

Fig. 1. Shows the scapula in a more perfect state than in the plate belonging to the former Paper; the spine, which in the other specimen was broken off, in this is in its place, although very nearly detached; it also shows the bone intermediate between the scapula and the small bones of the fin, with some of these bones on a scale of four inches to a foot. From a specimen of the Rev. Mr. BUCKLAND.

Fig. 2. The scapula and bones of the pectoral fin in the squalus acanthus, natural size.

PLATE XVI.

The bones of the pectoral fin, natural size. From a specimen of Mr. JOHNSON.

XXIII. *Farther observations on the feet of animals whose progressive motion can be carried on against gravity.* By Sir Everard Home, *Bart. V. P. R. S.*

Read June 27, 1816.

Since my observations on the foot of the lacerta gecko and fly were laid before the Society, Mr. Bauer, of Kew, has made drawings of the feet of both these animals; in the hands of an artist who has attained such excellence in the correct representation of objects highly magnified, the mechanism by which the feet are fitted for supporting the weight of the animal against gravity, is much better shown.

Mr. Bauer has not confined his labours to these objects, but has applied the powers of the microscope to the examination of the feet of other insects, and has enabled me to show, that the principle on which progressive motion against gravity depends, is very extensively employed by nature in the structure of the feet of insects; indeed the means employed for this purpose are so various, as to form characters by which many genera may be distinguished.

I shall not enter farther into this enquiry than to show some of the varieties of the structure.

My friend, Dr. Leach, whose researches in entymology fit him so admirably for the purpose, will, I trust, prosecute this subject, and ascertain the peculiarities that belong to the feet of different insects, fitting them for their peculiar habits

of life; which will assist him in giving a better arrangement of the subdivisions of this numerous class than has been hitherto done.

This structure of the feet of insects, now that it is known, can be very readily demonstrated by looking at the movements of the feet of any insect upon the inside of a glass tumbler, through a common magnifying glass; the different suckers are readily seen separately to be pulled off from the surface of the glass, and reapplied to another part.

The pockets on the under surface of the toes of the lacerta gecko, as they are represented in Plate XVII, show that what looked like a pectinated edge when seen through a common magnifying glass, consists of a complex structure, composed of rows of a beautiful fringe, which are applied to the surface on which the animal walks against gravity, while the pockets themselves are pulled up by the muscles attached to them, so as to form the cavities into suckers.

In the blue bottle fly, it will be seen, in Plate XVIII, that the suckers are two in number, that they are connected to the last joint of the toe, immediately under the root of the claw, and have a narrow infundibular neck attached to the toe, which has the power of motion in every direction; when these suckers are to be applied, they are separated from each other, and the membrane of each is expanded so as to increase the surface; but when disengaged, they become nearly closed, and are brought together, so as to be confined within the space between the two claws.

The external edge of each sucker is beautifully serrated, and the concave surface is granulated. When the fly is walking against gravity, and its motions are observed, they all appear

to be the result of muscular action, regulated by the will of the animal.

All the six toes are supplied with suckers.

In the horse fly, the tabanus of FABRICIUS, the suckers differ from those of the blue bottle fly in being three in number, in all other respects they are the same. In this fly, when the suckers are not used, the two outer ones close in before the other, and are only spread out when they are to be brought into use.

In the yellow Saw fly, the cimbex lutea of FABRICIUS, the suckers are differently situated from those of the fly; they are placed upon the under surface of the four first joints of the toes, one sucker upon each. These suckers are spoon shaped; they are represented in Plate XIX. The exterior part is thin and pellucid, but at half their depth they suddenly become thicker in their coats, forming a ridge at this part which gives the appearance of an inner cup, but this is a deception; the exterior membranous portion is alone expanded on the surface to which the sucker is applied, and the neck of the sucker forms the vacuum.

All the six feet have suckers.

The apparatus which has been described to support the animal in its progressive motion, is also applied to other purposes. In the great water beetle, the dytiscus marginalis, in which there is no appearance of suckers on the under side of the feet of the female, they are placed on the three first joints of the first and second pair of feet of the male, as is shown in Plate XX.; from which it is evident, that such suckers are used to retain the female in the embrace of the male. In the male, the three first joints of the feet of the

fore legs have the form of a shield, the under surface of which is covered with suckers, one very large, a second one-third smaller, and all the rest very small. In the second pair of feet, the corresponding joints are much narrower in proportion, and are covered on their under surface with very small suckers.

All these suckers, as is seen in Fig. 13, 14, and 15, have long tubular necks, which show more plainly than in the others the mode in which the vacuum is produced; it is exactly similar in its effect to that of a piece of leather with a string in the centre, applied in a moistened state to the surface of a stone.

Having explained this apparatus, so beautifully contrived to attach the feet to the surface on which the animal moves, I shall conclude this Paper by an account of a structure of a very different kind, for the purpose of taking off the jar when the body of the insect is suddenly brought from a state of motion to a state of rest; this is met with in the grylli and locustæ. Some of them have suckers at the ends of the toes, others have not.

In a species of gryllus with a corcelated thorax, brought from Abyssinia, by Mr. Salt, the feet are made up of three joints; on the under surface of the first are three pair of globular cushions, filled with an elastic fibrous substance, looser in its texture towards the circumference, which renders it still more elastic; under the second joint is one pair of similar cushions, and under the last joint, immediately between the claws, is a large oval sucker. A similar sucker is met with between the claws in a British grasshopper, the acrydium biguttulum (Latr.) These are common to all the six feet. They are represented in Plate XXI.

In the locusta varia, whose feet have four joints, under the first are two very small globular cushions, and two large oval ones; under the second, a corresponding pair of oval ones; and under the third, a pair of cushions different from the others, in being much larger, globular, and semi-transparent; there is no sucker between the claws, and this insect has no power of supporting itself against gravity.

As the flea has powers of jumping not exceeded by any other insect, it was natural to expect a similar apparatus under its feet; but as no such cushions are met with, we must conclude that the lightness of its body rendered them unnecessary.

EXPLANATION OF THE PLATES.

PLATE XVII.

Represents six different views of the third toe of the fore foot of the lacerta gecko.

Fig. 1. Is the upper surface of the toe, to show the manner in which it spreads laterally.

Fig. 2. The under surface of the same toe, to show the orifices of the pockets or suckers. These two figures are magnified 100 times.

Fig. 3. Two portions of two contiguous suckers, showing that the fringed termination is only continued from the ends of the alternate membranous partitions. The parts are magnified 2500 times.

Fig. 4. A front view of a longitudinal section, to show the bones and muscles.

Fig. 5. A side view of a longitudinal section. These two are magnified 100 times.

Fig. 6. A side view of a portion of some of the suckers, showing the insertion of the muscles, magnified 2500 times.

Plate XVIII.

Fig. 1. A left front leg of the blue bottle fly, musca vomitoria, magnified 100 times.

Fig. 2. A view of the under side of the last joint of the toe, with the two suckers expanded, as seen when the fly is walking against gravity.

Fig. 3. Side view of ditto.

Fig. 4. Upper side of ditto. These three figures are magnified 6400 times.

Fig. 5. View of the under side of a single sucker of a dead fly.

Fig. 6. Side view of ditto.

Fig. 7. Upper side of ditto. These three figures are magnified 6400 times.

Fig. 8. Left front leg of bibio febrilis (Latr.) Magnified 100 times.

Fig. 9. The under side of the last joint of the toe of ditto.

Fig. 10. Side view of ditto.

Fig. 11. Upper side of ditto. These three figures are magnified 6400 times.

Plate XIX.

Fig. 1. The left front leg of cimbex lutea, (Fabr.) *a*, thigh; *b*, shank; *c*, calces; *d*, toe; *e*, plantar suckers; *f*, sucker between the claws; *g*, unguis or claw.

Fig. 2. Side view of ditto.

Fig. 3. Under side of the left front leg. These three magnified 100 times.

Fig. 4. Last joint of toe, upper side, magnified 400 times.

Fig. 5. Under side of toe, and extremity of the shank, magnified 400 times.

Fig. 6 and 7. Upper and side views of two joints of the toe, magnified 400 times, to show the plantar suckers.

Fig. 8 and 9. Calces, magnified 1600 times, to show the form of the sucker with which each is terminated.

PLATE XX.

Fig. 1. The left front leg of the male dytiscus marginalis.

Fig. 2. Ditto, under side. They are both magnified 25 times.

Fig. 3. Tarsus of ditto.

Fig. 4. Side view of ditto.

Fig. 5. Under side of ditto. These three are magnified 100 times.

Fig. 6. Front view of the sucker (*a*) fig. 5, magnified 100 times.

Fig. 7. Side view of the sucker (*b*) fig. 5, magnified 100 times.

Fig. 8. Front view of the suckers (*c*) fig. 5, magnified 400 times.

Fig. 9. Several of the suckers (*c*) fig. 5, magnified 400 times.

Fig. 10. Second or middle left leg of dytiscus marginalis, magnified 25 times.

Fig. 11. Tarsus of ditto, under side, magnified 100 times.

Fig. 12. Several of the suckers (*a*) fig. 11, magnified 900 times.

Fig. 13, 14, and 15. Suckers (*a*) fig. 11. magnified 40,000 times, to show the articulation between the peduncle and the

sucker, and the joint by which the peduncle is attached to the tarsus.

Fig. 16. The hinder left leg of dytiscus marginalis, male, magnified 25 times.

Fig. 17. The left front leg of dytiscus marginalis, female, magnified 25 times.

Plate. XXI.

Fig. 1. The left front leg of a species of the genus gryllus, (Fabr.) with a corcelated thorax, from Abyssinia, magnified 9 times.

Fig. 2. A toe of ditto, to show the under side, on which are cushions attached to the first and second joints; *a*, the oval sucker between the claws; *bbbb*, the cushions.

Fig. 3. Ditto, side view.

Fig. 4. Ditto, upper side.

Fig. 5. Vertical section of the organs, fig. 2. *b*.

Fig. 6. Longitudinal section of ditto. All these are magnified 100 times.

Fig. 7. A front view of the left fore foot of a British species of grasshopper, acrydium biguttulum, (Latr.) to show that it has the same oval sucker between the claws, and the cushions, as in the grasshopper from Abyssinia. The parts were drawn from the animal while alive. The Abyssinian grasshopper had been preserved in spirit.

Fig. 8. A side view.

Fig. 9. A back view. These three views are magnified 2500 times.

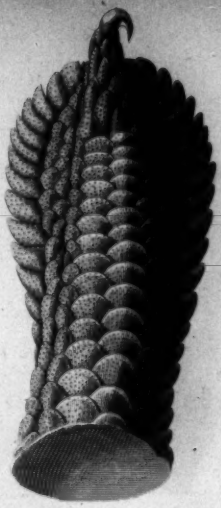
Fig. 10. A left front leg of a British species of grasshopper, locusta varia, (Fabr.) magnified 36 times.

Fig. 11. View of the underside of the toe.

Fig. 12. Side view of ditto.

Fig. 13. Upper side of the toe. These are magnified 625 times.

The cushions under the joints of the toe in this grasshopper, resemble in structure those of the Abyssinian gryllus, but differ in their form and situation. There is no sucker between the claws.



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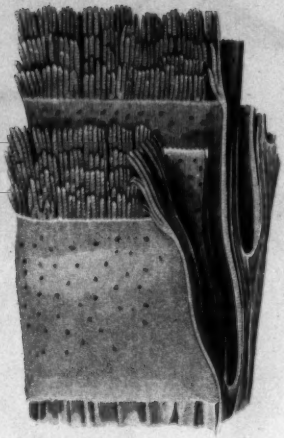

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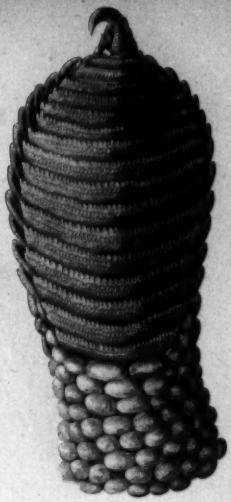

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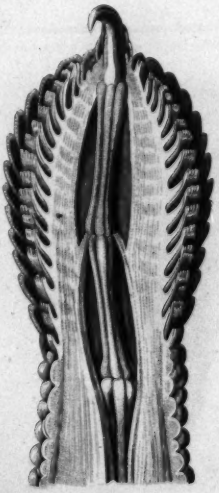

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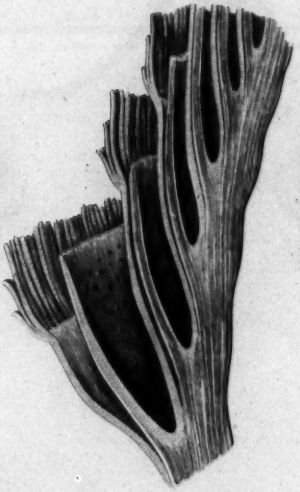

Fig. 6.

Fig. 5.

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Fig. 3

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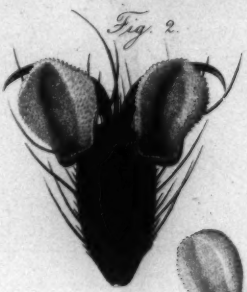

Fig. 4.

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Fig. 9 Fig. 10. Fig. 11.

Fig. 1.

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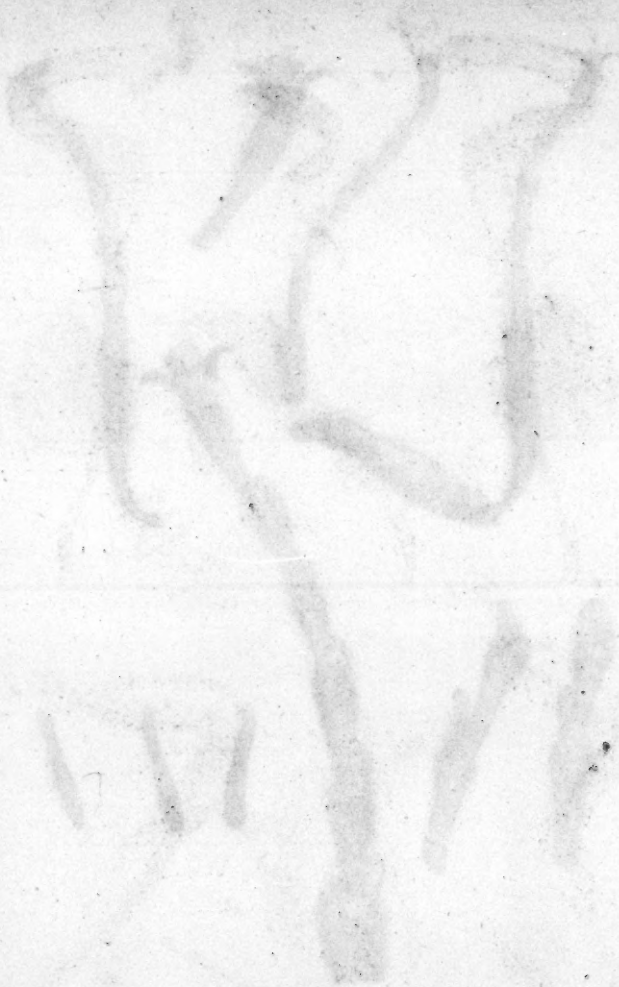

Fig. 1.

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Fig. 15.

Fig. 1.

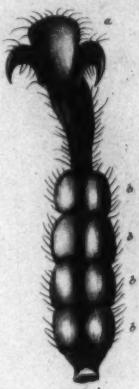

Fig. 2.

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Fig. 11.

Fig. 12.

Fig. 13.

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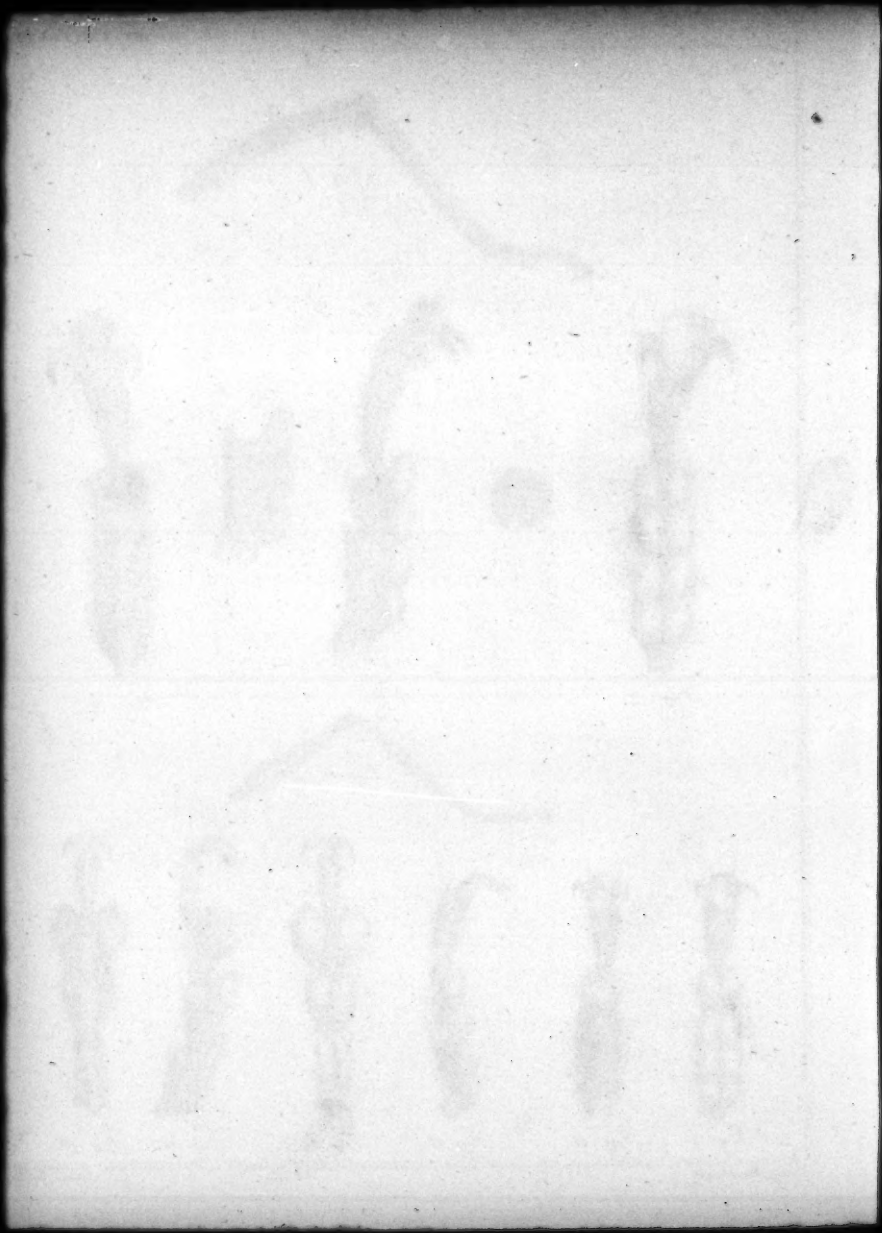

XXIV. *A new demonstration of the binomial theorem.* By Thomas Knight, *Esq.* *Communicated by* W. H. Wollaston, *M. D. Sec. R. S.*

Read July 4, 1816.

It is somewhat remarkable that, amongst the various and far-fetched methods and artifices by which the binomial theorem has been obtained, no one should once have thought of the only course which seems obvious and natural. The equation $(a+x)^m \times (a+y)^m = \{(a+x)(a+y)\}^m$ expresses the general property of powers, whether m be positive or negative, whole or fractional; and from this equation, without the help of any artifice, the series in question is deduced.

Some investigations have been found fault with, as drawn from principles allied to the method of fluxions; whilst, on the other hand, a demonstration, taken from the "*Théorie des Fonctions*," has been represented as perfect: but I cannot help thinking that it is as much connected with the fluxional calculus as any of the rest; for it seems to make no difference whether, in $(a+x)^m$, we substitute $x+u$ for x, and take the coefficient of u, or substitute $x+\dot{x}$, and take the coefficient of $\dot{x}$. The former substitution was made because it was *known* to be equivalent to the other, and has so little apparent connection with the subject, that a student would hardly understand why it was made. The demonstration of Mr. La Croix in the Introduction to his "*Calcul Différentiel*" is

liable of course to the same objection. If we multiply $a+x$, continually, first by itself, and then by the powers successively arising, we easily see that the second term of each succeeding product is of the form $na^{n-1}x$, n being the exponent of the power: this does not require a more formal proof, and I assume it in what follows. Nor is it more difficult to perceive that, generally, m being positive or negative, whole or fractional, the following form may be assumed,

$$(a+x)^m = a^m + {}'Aa^{m-1}x + {}''Aa^{m-2}x^2 + {}'''Aa^{m-3}x^3 +$$

where 'A, "A, "'A, &c. are expressions depending on m alone, consequently, $(a+y)^m = a^m + {}'Aa^{m-1}y + {}''Aa^{m-2}y^2 + {}'''Aa^{m-3}y^3 +$; and because $\{(a+x)(a+y)\}^m = (a^2 + ax + ay + xy)^m =$

$$= a^m\left(a+x+y+\frac{xy}{a}\right)^m = a^m(a+x+\pi y)^m, \text{ by making } \left(\pi = 1+\frac{x}{a}\right)$$

we have also

$$\{(a+x)(a+y)\}^m = a^m\{a^m + {}'Aa^{m-1}(x+\pi y) + {}''Aa^{m-2}(x+\pi y)^2 + {}'''Aa^{m-3}(x+\pi y)^3 + \}$$

or neglecting all the powers of y but the first,

$$= a^m\{a^m + {}'Aa^{m-1}x + {}''Aa^{m-2}x^2 + {}'''Aa^{m-3}x^3 + \}$$
$$+ a^m y\{{}'Aa^{m-1}\pi + 2\,{}''Aa^{m-2}\pi x + 3\,{}'''Aa^{m-3}\pi x^2 + \}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Having thus the forms of the series, nothing is required but to substitute them in the equation

$$(a+x)^m \times (a+y)^m = \{(a+x)(a+y)\}^m$$

and to compare the coefficients of the first power of y on each side, and we find

$${}'Aa^{2m-1} + {}'A\,{}'Aa^{2m-2}x + {}'A\,{}''Aa^{2m-3}x^2 + {}'A\,{}'''Aa^{2m-4}x^3 +$$
$$= {}'Aa^{2m-1}\pi + 2\,{}''Aa^{2m-2}\pi x + 3\,{}'''Aa^{2m-3}\pi x^2 + 4\,{}''''Aa^{2m-4}\pi x^3 +$$

$$= {}'Aa^{2m-1} + 2\,{}''Aa^{2m-2} \left.\begin{matrix} \\ + {}'Aa^{2m-2} \end{matrix}\right\} x \left.\begin{matrix} + 3\,{}'''Aa^{2m-3} \\ + 2\,{}''Aa^{2m-3} \end{matrix}\right\} x^2 \left.\begin{matrix} + 4\,{}''''Aa^{2m-4} \\ + 3\,{}'''Aa^{2m-4} \end{matrix}\right\} x^3 +$$

by putting for π its value $1 + \frac{x}{a}$. And by comparing the coefficients of the different powers of x there arise ${}'A = {}'A$; ${}''A = 2\,{}''A + {}'A$; ${}'A\,{}''A = 3\,{}'''A + 2\,{}''A$; ${}'A\,{}'''A = 4\,{}''''A + 3\,{}'''A$; and so on; whence ${}''A = \frac{{}'A({}'A-1)}{2}$; ${}'''A = \frac{{}''A({}'A-2)}{3}$; ${}''''A = \frac{{}'''A({}'A-3)}{4}$; and so on.

Such is the law by which the coefficients are derived from each other, whatever be the value of m; it remains to find ${}'A$; but I shall first observe, that if, instead of the assumed form of the expansion, we had made $(a+x)^m = a^m + {}'Aa^{m-1}x + {}''Ax^2 + {}'''Ax^3 +$ as some do, our demonstration would have succeeded exactly the same; because the exponent (m) and the first term (a) of the binomials are the same in all the three powers employed.

We have already seen that ${}'A = m$, if m be a whole positive number; or that $(a+x)^m = a^m + ma^{m-1}x +$; and from the value of ${}'A$ in this one case its value in all the others is easily discovered: thus, let

$(a+x)^{\frac{1}{m}} = a^{\frac{1}{m}} + {}'Aa^{\frac{1}{m}-1}x +$, the m^{th} power of this is $a+x$;

but $\left(a^{\frac{1}{m}} + {}'Aa^{\frac{1}{m}-1}x + \right)^m = a^{\frac{m}{m}} + m\,{}'Aa^{\frac{m-1}{m}} \times a^{\frac{1}{m}-1}x + = a + m\,{}'Ax +$

but $a + m\,{}'Ax + = a + x$, consequently ${}'A = \frac{1}{m}$; and

$(a+x)^{\frac{1}{m}} = a^{\frac{1}{m}} + \frac{1}{m}a^{\frac{1-m}{m}}x +$, next raise this to the n^{th} power, n being a positive integer,

$$(a+x)^{\frac{n}{m}} = \left(a^{\frac{1}{m}} + \frac{1}{m}a^{\frac{1-m}{m}}x + \right)^n = a^{\frac{n}{m}} + na^{\frac{n-1}{m}} \times \frac{1}{m}a^{\frac{1-m}{m}}x +$$

$= a^{\frac{n}{m}} + \frac{n}{m} a^{\frac{n}{m}-1} x +$. Lastly $(a+x)^{\frac{-n}{m}} = \frac{1}{(a+x)^{\frac{n}{m}}} =$

$= \frac{1}{a^{\frac{n}{m}} + \frac{n}{m} a^{\frac{n}{m}-1} x +}$ = (by actual division) $a^{\frac{-n}{m}} - \frac{n}{m} a^{\frac{-n}{m}-1} x +$

In all cases therefore 'A is equal to the exponent of the power.

XXV. *On the fluents of irrational functions.* *By* Edward Ffrench Bromhead, *Esq. M. A. Communicated by* J. F. W. Herschel, *Esq. F. R. S.*

Read June 4, 1816.

THE efforts of analysts in determining the fluents of rational functions, have been completely successful, and their labours form one of the most perfect and beautiful branches of the fluxionary calculus. In the irrational functions, however, we find but little effected. With the exception of WARING, modern analysts have not added any thing important, to the forms given by NEWTON, CRAIG, COTES, and BERNOULLI. No attempt has been made to generalize the known forms, and the last eminent writer on the subject, LA CROIX, seems to consider them as independent results, not deducible from any common principles, and refers us to the Petersburgh Acts, and other miscellaneous Collections. In the following pages, it is attempted to generalize and systematize our knowledge on this subject; and to show that all the known forms result from other forms of the greatest extent, not depending on particular functions, but upon the properties of all rational functions whatever.

$R, \dot{R}, \ddot{R}, \dot{\ddot{R}}$, denote rational functions of any kind; R^{-1}, $\dot{R}^{-1}$, $\ddot{R}^{-1}$, $\dot{\ddot{R}}^{-1}$ their inverse functions. Thus if $x = R(v)$ any rational function of (v), then $v = R^{-1}(x)$ the inverse function.

It is thought unnecessary to prove, that the fluxions of all rational functions, and all rational functions of them, are themselves rational.

PROP. I.

$dx \,.\, \mathrm{R}\left\{x, \mathrm{R}^{-1}(x)\right\}$ can be rationalized.

Let $\mathrm{R}^{-1}(x) = v$; $x = \mathrm{R}(v)$; $dx = dv \,.\, \mathrm{DR}(v)$*

which substituted produce the rational form

$$dv \,.\, \mathrm{DR}(v) \,.\, \mathrm{R}\left\{\mathrm{R}(v), v\right\}$$

Cor. 1. This form includes

$$dx \,.\, \mathrm{R}\left\{x, \mathrm{R}^{-1}(x), \mathrm{R}\left(x, \mathrm{R}^{-1}(x)\right), \ldots \mathrm{R}\left(x, \mathrm{R}^{-1}(x)\right)\right\}$$

Cor. 2. We may find, a priori, what fluents will come under this form. For let $x = \mathrm{R}(v)$ any rational function whatever

$$= \frac{\alpha + \beta \,.\, v + \gamma \,.\, v^2 + \ldots}{\alpha + \alpha' \,.\, v + \alpha'' \,.\, v^2 + \ldots}$$

which is the general form taken by rational functions, when the integer powers are expanded, and the fractions reduced to a common denominator, the coefficients being positive, negative, or nothing. Hence we have $\mathrm{R}^{-1}(x) = v$, determined from this equation.

$$\left(\alpha \,.\, x - \alpha\right) + \left(\beta \,.\, x - \alpha'\right) \,.\, v + \left(\gamma \,.\, x - \alpha''\right) \,.\, v^2 + \ldots = 0$$

Cor. 3. Let the equation be

$$(ax - \alpha) + (bx - \beta)\, v^n = 0$$

Then $\mathrm{R}^{-1}(x) = v = \left\{\frac{-ax + \alpha}{bx - \beta}\right\}^{\frac{1}{n}}$

$\therefore$ we know $\int dx \,.\, \mathrm{R}\left\{x, \left(\frac{ax + \alpha}{bx - \beta}\right)^{\frac{1}{n}}\right\}$

* See Note at the end of the Paper.

Cor. 4. Let the equation be

$$(ax-\dot{a})+(b\dot{x}-\beta)\,.\,v^{n}+(cx-\gamma)\,.\,v^{2n}=0$$

from which we can determine

$$\int dx\,.\,\mathrm{R}\left\{x,\left(-\frac{bx-\beta}{2cx-2\gamma}\pm\sqrt{\left(\frac{bx-\beta}{2cx-2\gamma}\right)^{2}-\frac{ax-a}{cx-\gamma}}\right)^{\frac{1}{n}}\right\}$$

and therefore any fluent of the form

$$\int dx\,.\,\mathrm{R}\left\{x,\sqrt[n]{ax+\delta\pm\sqrt{a^{2}x^{2}+\beta x+\gamma}}\right\}$$

It is obvious that these deductions may be carried to any extent, producing forms hitherto supposed impracticable.

Cor. 5. $\frac{ax+\beta}{ax+b}$ is both of the form $\mathrm{R}^{-1}(x)$ and $\mathrm{R}^{+1}(x)$.

PROP. II.

We can rationalize

$$dx\,.\,\mathrm{R}\left\{x,\mathrm{R}^{-1}(x),\mathrm{R}^{-1}\mathrm{R}^{-1}(x),\ldots\mathrm{R}^{-1}\mathrm{R}^{-1}\ldots\mathrm{R}^{-1}\mathrm{R}^{-1}(x)\right\}$$

Let $\mathrm{R}^{-1}\ldots\mathrm{R}^{-1}(x)=v$. Then

$$x=\mathrm{R}\,\mathrm{R}\ldots\mathrm{R}\,(v)$$

$$\mathrm{R}^{-1}x=\mathrm{R}\ldots\mathrm{R}\,(v)$$

$$\&c.\;=\;\&c.$$

which substituted in the original expression, make it rational.

Cor. 1. If $\mathrm{R}=\mathrm{R}=\mathrm{R}$, the fluxion becomes

$$dx\,.\,\mathrm{R}\left\{x,\mathrm{R}^{-1}(x),\mathrm{R}^{-2}(x),\ldots\mathrm{R}^{-n}(x)\right\}$$

Cor. 2. By this theorem, any of the expressions deducible from Prop. I. may enter contemporaneously, and we may find fluents of very great intricacy.

Cor 3. The fluents

$$\int dx\,.\,\mathrm{R}\left\{x,(a+\beta x)^{\frac{1}{n}},\left(\frac{a+b\sqrt[n]{a+\beta x}}{c+d\sqrt[n]{a+\beta x}}\right)^{\frac{1}{m}},\ldots\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt[n]{a + \sqrt[m]{b + \sqrt[r]{\ldots \sqrt[p]{\mathrm{R}^{-1}(x)}, \ldots}}}\right\}$$

and some of the most complex expressions in WARING's Med. Anal. are very particular cases of this form.

PROP. III.

We can rationalize

$$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}^{-1}}(x), \left(\underset{1}{\mathrm{R}^{-1}}(x)\right)^{\frac{1}{m}}, \left(\underset{1}{\mathrm{R}^{-1}}(x)\right)^{\frac{1}{n}}, \left(\underset{1}{\mathrm{R}^{-1}}(x)\right)^{\frac{1}{r}}, \ldots\right\}$$

Let $\underset{1}{\mathrm{R}^{-1}}(x) = v^{m \cdot n \cdot r \cdots}$

Then $x = \underset{1}{\mathrm{R}}(v^{m \cdot n \cdot r \cdots})$

$$\left\{\underset{1}{\mathrm{R}^{-1}}(x)\right\}^{\frac{1}{m}} = v^{n \cdot r \cdots}$$

$$\left\{\underset{1}{\mathrm{R}^{-1}}(x)\right\}^{\frac{1}{n}} = v^{m \cdot r \cdots}$$

&c. = &c.

which substituted make the expression rational.

Cor. 1. The more general form is this:

If $\underset{m}{\mathrm{R}}$ can be so assumed that $\underset{1}{\mathrm{R}^{-1}}\underset{m}{\mathrm{R}}$, $\underset{2}{\mathrm{R}^{-1}}\underset{m}{\mathrm{R}}$, $\underset{3}{\mathrm{R}^{-1}}\underset{m}{\mathrm{R}}$ shall be all rational; then by assuming $\underset{4}{\mathrm{R}^{-1}}(x) = \underset{m}{\mathrm{R}}(v)$ we can render rational

$$dx \,.\, \mathrm{R}\left\{x, \underset{2}{\mathrm{R}^{-1}}(x), \underset{3}{\mathrm{R}^{-1}}\underset{2}{\mathrm{R}^{-1}}(x), \ldots \underset{m}{\mathrm{R}^{-1}}\underset{3}{\mathrm{R}^{-1}}(x)\right\}$$

Cor. 2. We can find

$$\int dx \,.\, \mathrm{R}\left\{x, x^{\frac{1}{m}}, x^{\frac{1}{n}}, x^{\frac{1}{r}}, \ldots\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, (ax + \beta)^{\frac{1}{m}}, (ax + \beta)^{\frac{1}{n}}, \ldots\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, \left(\frac{ax + \beta}{ax + b}\right)^{\frac{1}{m}}, \left(\frac{ax + \beta}{ax + b}\right)^{\frac{1}{n}}, \ldots\right\}$$

$\int dx \,.\, \mathrm{R}\left\{x, \sqrt{\alpha^2 x^2 + \beta x + \gamma}, (\alpha x + \sqrt{\alpha^2 x^2 + \beta x + \gamma})^{\frac{2}{m}}, \ldots\ldots\right\}$

with an indefinite number of forms too complex for convenient expression.

Cor. 3. This form may be extended to Prop. II. and other general expressions. Thus we know

$$\int dx . \mathrm{R}\left\{x, \mathrm{R}_{2}^{-1}(x), \mathrm{R}_{3}^{-1}\mathrm{R}_{1}^{-1}(x), \ldots . (\mathrm{R}_{n}^{-1} .. \mathrm{R}_{1}^{-1}(x))^{\frac{1}{m}}, (\mathrm{R}_{n}^{-1} \ldots \mathrm{R}_{1}^{-1}(x))^{\frac{1}{n}}, \ldots\right\}$$

The forms given above are wholly inapplicable, when the fluxion involves expressions, such as $\mathrm{R}_{3}^{-1}\mathrm{R}_{2}\mathrm{R}_{1}^{-1} \ldots (x)$ where the functions are alternately inverse and direct. The cases are very few, in which the difficulty can be overcome, and perhaps the following Propositions will be found to include all the instances, in which analysts have effected the reduction.

PROP. IV.

We can rationalize

$$dx \,.\, \mathrm{DR}_{2}(x) \,.\, \left\{\mathrm{R}_{2}(x), \mathrm{R}_{3}^{-1}\mathrm{R}_{2}(x)\right\}$$

Let $\mathrm{R}_{2}(x) = v$, and it becomes

$$dv \,.\, \mathrm{R}\left\{v, \mathrm{R}_{3}^{-1}(v)\right\} \text{ as in Prop. 1.}$$

Cor. 1. We can generally reduce $\int dx \,.\, \mathrm{D}\phi(x) \,.\, \phi_{2}\phi(x)$ to $\int dv \,.\, \phi_{2}(v)$. Thus we deduce from $\int dx \,.\, x^{rn-1} \,.\, \phi(x^n)$ the $\int dv \,.\, v^{r-1}\phi(v)$, and from $\int \frac{dx}{x} \,.\, \phi(x^n)$ the $\int \frac{dv}{v} \,.\, \phi(v)$, reductions of frequent occurrence, by which analysts have given their forms an appearance of generalization without the reality.

Cor. 2. This form may be extended to all the former Propositions.

Cor. 3. As it is very tedious and often impracticable to find x in terms of v, in order to know whether the reduction be applicable; the following process may sometimes be useful. Let the expression be

$$dx \,.\, \mathrm{DR}_{\circ}(x) \,.\, \mathrm{R}_{\circ}^{-1}\mathrm{R}_{\circ}(x) \,.\, \mathrm{R}_{\circ}\mathrm{R}_{\circ}(x)$$

Then if it be divided by $dx \,.\, \mathrm{DR}_{\circ}(x) \,.\, \mathrm{R}_{\circ}^{-1}\mathrm{R}_{\circ}(x)$ the quotient will be a rational function of $\mathrm{R}_{\circ}(x)$ or of the form

$$\frac{a_{\circ} + a_{\circ} \,.\, \mathrm{R}_{\circ}(x) + a_{\circ} \,.\, (\mathrm{R}_{\circ}(x))^{2} + \ldots}{a_{\circ} + a_{\circ} \,.\, \mathrm{R}_{\circ}(x) + a_{\circ} \,.\, (\mathrm{R}_{\circ}(x))^{2} + \ldots}$$ the coefficients being indeterminate. If the reduction be applicable, these may be found, and the substitution made at once.

Cor. 4. We may thus reduce

$$\int dx \,.\, \frac{nx^{n-1} + (n-1)\,c_{1} \,.\, x^{n-2}}{m\sqrt{x^{n} + c_{\circ}x^{n-1} + \ldots}} \,.\, \mathrm{R}\left\{x^{n} + c_{1}x^{n-1} + \ldots\right\}$$

to $\int \frac{dv}{\frac{1}{m}} \,.\, \mathrm{R}\,(v)$, which may be found.

Cor. 5. In $dx \,.\, x^{rn-1} \,.\, (ax^{n} + \beta)^{\mu} \,.\, \mathrm{R}\left\{ax^{n} + \beta\right\}$ divide by $dx \,.\, nax^{n-1} \,.\, (ax^{n} + \beta)^{\mu}$, and the quotient is $\frac{1}{na} \,.\, x^{(r-1)n} \,.\, \mathrm{R}\,(ax^{n} + \beta)$. The latter factor is already of the required form, and by assuming

$$x^{(r-1)n} = a_{\circ} \,.\, (ax^{n} + \beta)^{r-1} + a_{\circ} \,.\, (ax^{n} + \beta)^{r-2} + \ldots$$

the indeterminates may be found. In particular cases there are readier processes, but this method is universally applicable.

PROP. V.

We can rationalize

$dx \cdot \mathrm{R}\left\{x, \overset{n}{\mathrm{R}}{}^{-1}(x)\right\}$ if we can determine, $\overset{n}{\mathrm{R}}{}^{-1}(x) = \overset{\mu}{\mathrm{R}}(x) \cdot \overset{\nu}{\mathrm{R}}{}^{-1}(x)$, for the fluxion then becomes

$dx \cdot \mathrm{R}\left\{x, \overset{\mu}{\mathrm{R}}(x) \cdot \overset{\nu}{\mathrm{R}}{}^{-1}(x)\right\}$ as in Prop. I. Cor. 1.

Cor. 1. We also know

$$dx \cdot \mathrm{R}\left\{x, \overset{\nu}{\mathrm{R}}{}^{-1}(x), \overset{n}{\mathrm{R}}{}^{-1}(x)\right\}$$

Cor. 2. We may thus transform

$$dx \cdot \mathrm{R}\left\{x, \left(\frac{ax+\beta}{ax+b}\right)^{\frac{\nu}{\mu}}, \sqrt[n]{(ax+\beta)\cdot(ax+b)^{n-1}}\right\}$$

into

$$dx \cdot \mathrm{R}\left\{x, \left(\frac{ax+\beta}{ax+b}\right)^{\frac{1}{n}}, (ax+b)\cdot\left(\frac{ax+\beta}{ax+b}\right)^{\frac{1}{n}}\right\}$$

a known form.

Hence we know

$$\int dx \cdot \mathrm{R}\left\{x, \sqrt[3]{(\alpha^3 x^3 - \beta^3)\cdot(ax \pm \beta)}\right\}$$

$$\int dx \cdot \mathrm{R}\left\{x, \sqrt[3]{(x+\alpha)\cdot(x+\beta)^2}\right\}$$

$$\int dx \cdot \mathrm{R}\left\{x, \sqrt{(ax+\beta)\cdot(ax+b)}\right\}$$

$$\int dx \cdot \mathrm{R}\left\{x, \sqrt{(cx^2+c\,x+c}\right\}$$

which last form will sometimes introduce imaginaries, that may be avoided by particular artifices.

Cor. 3. If $\overset{n}{\mathrm{R}}{}^{-1}(x) = \left(\overset{\mu}{\mathrm{R}}(x)\right)^n \cdot \overset{\nu}{\mathrm{R}}{}^{-1}(x)$

$\overset{n}{\mathrm{R}}{}^{-1}(x) = \left\{\overset{\mu}{\mathrm{R}}(x)\right\}^n \cdot \overset{\nu}{\mathrm{R}}{}^{-1}(x)$

&c. = &c.

we can determine

$$\int dx \,.\, \mathrm{R}\left\{x, \underset{2}{\mathrm{R}}^{-1}(x), \left\{\underset{1}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}(x)\right\}^{\frac{1}{m}}, \left\{\underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x)\right\}^{\frac{1}{n}}, \ldots\right\}$$

Cor. 4. If $\underset{1}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}(x) = \underset{\mu}{\mathrm{R}}(x)\sqrt[m]{\underset{\alpha}{\mathrm{R}}^{-1}(x)}$; $\underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x) = \underset{\nu}{\mathrm{R}}(x)\cdot\sqrt[n]{\underset{\alpha}{\mathrm{R}}^{-1}(x)}$; &c. = &c.,

we know

$$\int dx \,.\, \mathrm{R}\left\{x, \underset{2}{\mathrm{R}}^{-1}(x), \underset{1}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}(x), \underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x), \underset{1}{\mathrm{R}}^{-1}\underset{r}{\mathrm{R}}(x), \ldots\right\}$$

as in

$$dx \,.\, \mathrm{R}\left\{x, \sqrt[m]{(ax+\beta).(ax+b)^{m-1}}, \sqrt[n]{(ax+\beta).(ax+b)^{n-1}}, \ldots\right\}$$

Cor. 5. Generally if $\underset{1}{\mathrm{R}}(x)$ and $\underset{2}{\mathrm{R}}(x)$ are so related, that $\underset{1}{\mathrm{R}}\,\underset{2}{\mathrm{R}}(x)$ can $= \underset{2}{\mathrm{R}}\,\underset{m}{\mathrm{R}}(x)$, $\underset{m}{\mathrm{R}}$ and $\underset{m}{\mathrm{R}}$ being any rational functions whatever taken at pleasure, then $dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}\underset{2}{\mathrm{R}}(x)\right\}$ can be rationalized by taking $x = \underset{m}{\mathrm{R}}(v)$. It then becomes

$$dv \,.\, \mathrm{DR}(v) \,.\, \mathrm{R}\left\{\underset{m}{\mathrm{R}}(v), \underset{2}{\mathrm{R}}(v)\right\}.$$

PROP. VI.

By combining Prop. IV. and V, we can rationalize

$$dx \,.\, \mathrm{DR}(x) \,.\, \mathrm{R}\left\{\underset{m}{\mathrm{R}}(x), \underset{1}{\mathrm{R}}^{-1}\,\underset{n}{\mathrm{R}}\,\underset{m}{\mathrm{R}}(x)\right\}$$

if $\underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(v) = \underset{\nu}{\mathrm{R}}(v) \,.\, \underset{2}{\mathrm{R}}^{-1}(v)$; for let $\underset{m}{\mathrm{R}}(x) = v$, and it becomes

$$dv \,.\, \mathrm{R}\left\{v, \underset{\nu}{\mathrm{R}}(v) \,.\, \underset{2}{\mathrm{R}}^{-1}(v)\right\} \text{ as before.}$$

Cor. 1. If we have

$$dx \,.\, x^{m-1}(ax^n+\beta)^{\frac{p}{q}} \,.\, \mathrm{R}\left\{a+\beta x^{-n}\right\}$$

Remove the multiplier x^n, as in Prop. V., and it becomes,

$$\frac{dx}{x} \cdot x^{m+\frac{np}{q}} \cdot (\alpha + \beta x^{-n})^{\frac{p}{q}} \cdot \mathrm{R}(\alpha + \beta x^{-n})$$

which will fall under Prop. IV. If $x^{m+\frac{np}{q}}$ can be expressed by a rational function of $\alpha + \beta x^{-n}$. This will happen if $m + \frac{np}{q} = -n \cdot r$, or if $\frac{m}{n} + \frac{p}{q} = \pm r$ any integer. Hence we know

$$\int \frac{dx}{(ax^{\alpha} + bx^{\beta})^{\frac{1}{n}}} \cdot \mathrm{R}\left\{a + bx^{\beta-\alpha}\right\}$$

$$\int \frac{dx}{\sqrt[3]{\alpha + \beta x^3}}; \int \frac{dx}{\sqrt[3]{x^3 + 1}}; \int \frac{dx}{\sqrt[4]{x^4 - 1}};$$

Cor. 2. We can determine

$$\int \frac{ax^{2n} - \beta}{ax^{2n} + \gamma x^n + \beta} \cdot \frac{dx}{\sqrt[n]{ax^{2n} + cx^n + \beta}}$$

which becomes by Prop. 5.

$$\frac{ax^{n} - \beta x^{-n-1}}{ax^{n} + \beta x^{-n} + \gamma} \times \cdot \frac{dx}{\sqrt[n]{ax^n + \beta x^{-n} + c}}$$

Now this falls under Prop. IV. For let

$$\sqrt[n]{ax^n + \beta x^{-n} + c} = v$$

$$ax^n + \beta x^{-n} + \gamma = v^n + \gamma - c$$

$$\therefore (ax^{n-1} - \beta x^{-n-1}) \cdot dx = v^{n-1} \cdot dv$$

Whence the fluxion becomes $\frac{v^{n-2} \cdot dv}{v^n + \gamma - c}$, of which a particular case is deduced in LEGENDRE's Elliptic Transcendents.

PROP. VII.

If we can rationalize

$$dx \cdot \mathrm{R}\left\{x, \varphi(x), \varphi_{\cdot}(x), \ldots\right\}$$

we also can

$$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x), \underset{2}{\mathrm{R}}^{-1}(x), \underset{3}{\mathrm{R}}^{-1}(x), \ldots\right\}$$

for by taking $\underset{1}{\mathrm{R}}^{-1}(x) = v$, it is reduced to the former form.

Cor. 1. If we can rationalize

$$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}\underset{2}{\mathrm{R}}(x)\right\}, \text{ we also can}$$

$$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x), \underset{2}{\mathrm{R}}^{-1}\underset{3}{\mathrm{R}}\underset{4}{\mathrm{R}}^{-1}(x)\right\}$$

Therefore we can find

$$\int dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x), \sqrt{a + b \,.\, \underset{2}{\mathrm{R}}^{-1}(x) + c \,.\, (\underset{3}{\mathrm{R}}^{-1}(x))^2}\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt{\alpha + \beta x + \sqrt{a + bx}}\right\} \text{ &c.}$$

Cor. 2. Generally we can reduce

$$dx \,.\, \mathrm{R}\left\{x, \underset{0}{\mathrm{R}}\underset{1}{\mathrm{R}}^{-1} \ldots \underset{n}{\mathrm{R}}\underset{n+1}{\mathrm{R}}^{-1}(x)\right\} \text{ to}$$

$$dx \,.\, \mathrm{R}\left\{x, \underset{a}{\mathrm{R}}^{-1} \ldots \underset{a}{\mathrm{R}}(x)\right\}$$

Cor. 3. In $\int dx \,.\, \varphi(x)$, let $v = \varphi(x)$, and if it be an algebraic function, $\mathrm{R}\{x, v,\} = o$. Now take $x = \underset{a}{\mathrm{R}}(z) = \frac{\alpha + \underset{1}{a}z + \underset{2}{a}z^2 + \text{&c.}}{\underset{0}{a} + \underset{1}{a}z + \underset{2}{a}z^2 + \text{&c.}}$; and $v = \underset{a}{\mathrm{R}}(z) = \frac{\beta + \underset{1}{\beta}z + \text{&c.}}{\underset{0}{b} + \underset{1}{b}z + \text{&c.}}$ with indeterminate coefficients.

Hence we have $\mathrm{R}\left\{\underset{1}{\mathrm{R}}(z), \underset{2}{\mathrm{R}}(z)\right\} = o$; remove fractions and make the coefficients of the powers of z vanish. This will give the indeterminates, if x and v admit common rationalities. Thence we have $dz \,.\, \mathrm{DR}(z) \,.\, \underset{2}{\mathrm{R}}(z)$ rational.

Should all the artifices in the foregoing propositions fail, we must attempt to resolve the fluxion into a series of terms, such that each term may be separately rationalized. This is

often possible, when the original function does not admit a rational expression, and can be effected sometimes directly, and sometimes by introducing a new variable. But it will first be necessary to reduce all irrational functions whatever to a definite form.

LEMMA.

To reduce all irrational functions to a definite form.

1. By successively multiplying numerators and denominators into the same expressions, every irrational function may at last be reduced to a series of terms, whose numerators and denominators do not contain any fraction or negative index.

Thus $\frac{\left(\alpha + \frac{\beta}{\gamma}\right)^{\mu}}{(a + bc^{-n})^{m}} = \frac{c^{nm} \cdot (\alpha\gamma + \beta)^{\mu}}{\gamma^{\mu} \cdot (ac^{n} + b)^{m}}$, and if $\alpha, \beta, \gamma, a, b, c$, are functions involving fractions or negative indices, themselves, the reduction is continued in the same manner.

2. Now multiply both the numerators and denominators of the expressions so reduced, by such multipliers, as will render the denominators rational. This factor is the product of all the different values of the denominator, with the exception of the denominator itself. The new numerators will still consist of a series of terms not involving any fraction or negative index.

3. If R, R_1, R_2, &c. denote functions of the form $cx^{m} + c_1 x^{n-1} + \ldots$, the irrational takes the form,

$$\frac{R(x)}{R_1(x)} + \frac{R_2(x)}{R_3(x)} \cdot \varphi_1(x)^{\frac{a}{b}} \cdot \varphi_2(x)^{\frac{c}{d}} \ldots + \frac{R_4(x)}{R_5(x)} \cdot \pi_1(x)^{\frac{e}{\beta}} \ldots + \ldots$$

4. By reducing the fractional indices of the factors to the

common denominator (n), the whole will consist of a series of terms $\frac{\overset{R}{\underset{s}{}}(x)}{\overset{R}{\underset{s}{}}(x)} \cdot \sqrt[n]{\overset{\varphi}{\underset{1}{}}(x)_p \cdot \overset{\varphi}{\underset{2}{}}(x)^q \ldots}$

5. By expanding all the integer powers under the index $\frac{1}{n}$; and again reducing the indices of the sums and products, which are under it, to a common denominator n'; we shall by continuing the same operations, ultimately reduce the whole expression, to a series of terms of the form

$$\frac{\overset{R}{\underset{s}{}}(x)}{\overset{R}{\underset{s}{}}(x)} \sqrt[n]{S^{n'}\sqrt{S^{n''}\sqrt{\ldots S^{n^{(r)}}\ldots}\sqrt{\overset{R}{\underset{m}{}}(x)}}}$$

S denoting the sum of any number of terms such as follow it, wherein $\overset{R}{\underset{m}{}}(x)$ may be different in each term, but always of the form $cx^m + \overset{c}{\underset{s}{}}x^{m-1} + \ldots +$.

6. If *every* value of $\overset{R}{\underset{m}{}}(x)$ contains a factor $(ax^r + bx^{r-1} + \ldots)^{n \cdot n' \cdot n'' \ldots}$, it may be taken entirely out of the radical; and conversely the rational coefficient may be introduced entirely under the radical.

7. When the surd is so reduced, that no rational factor can be withdrawn from the radical, it is said to be in its lowest terms; and is said to be an irrational of the 1^{st}, 2^{d}, or r^{th} order, according to the number of the indices $\frac{1}{n}$, $\frac{1}{n'}$, $\frac{1}{n^{(r-1)}}$. Thus the general expression for a surd of the first order

is a series of terms, $\frac{\overset{R}{\underset{s}{}}(x)}{\overset{R}{\underset{s}{}}(x)} \sqrt[n]{cx^m + \overset{c}{\underset{1}{}}x^{m-1} + \ldots}$

8. A more convenient general form for all irrationals, than

the series of terms above exhibited, may readily be found; by introducing all the rational parts entirely under the radicals; by reducing the indices of all the terms to a common denominator μ; by expanding all integer powers; and by again reducing all the products and sums contained under $\frac{1}{\mu}$, to indices with a common denominator μ'. These operations continued, will ultimately lead to the expression

$$S\sqrt[\mu]{\;S\sqrt[\mu']{\;\ldots S\sqrt[\mu'\ldots]{\frac{R(x)}{\frac{\alpha}{R(x)}}}}}_{\beta}, \text{ where } \frac{R(x)}{\frac{\alpha}{R(x)}}_{\beta} \text{ may be of any}$$

different values in the different sums, but always of the form

$$\overline{ax^{\alpha} + \overset{\cdot}{a}x^{\alpha-1} + \ldots}$$
$$\overline{bx^{\beta} + \overset{\cdot}{b}x^{\beta-1} + \ldots}$$
$$\overset{R(x)}{\cdot}$$

9. $\frac{\alpha}{R(x)}_{\beta}$ is said to be of $\alpha - \beta$ dimensions; and if $\alpha - \beta$ be dimensions of that rational part, whose dimensions are greatest; then the dimensions of the whole irrational are $\frac{\alpha-\beta}{\mu \cdot \mu' \cdot \mu'' \ldots}$

10. The fluxion, and its dimensions in any irrational, may be found by applying this formula, $d\{\overset{\varphi}{\cdot}\overset{\varphi}{\cdot}\overset{\varphi}{\cdot}\ldots\overset{\varphi}{\cdot}(x)\} = D\overset{\varphi}{\cdot}\overset{\varphi}{\cdot}\ldots\overset{\varphi}{\cdot}(x) \cdot D\overset{\varphi}{\cdot}\overset{\varphi}{\cdot}\ldots\overset{\varphi}{\cdot}(x)\ldots D\overset{\varphi}{\cdot}(x) \cdot dx$, the D only referring to the functional characteristic immediately succeeding it.

PROP. VIII.

To divide a fluxion into expressions admitting distinct rationalities.

Let $\phi(x)$ be any irrational, and $\phi_1(x)$, $\phi_2(x)$ &c. surds deduced as in the Lemma. Then

$$\int dx \cdot \phi(x) = \int \frac{dx \cdot \mathrm{R}\ x)}{\mathrm{R}_1(x)} + \int \frac{dx \cdot \mathrm{R}(x)}{\mathrm{R}_2(x)} \cdot \phi_1(x) + \text{&c.}$$

where the fluent of the 1st term may always be found, and the other terms may often be rationalized by distinct substitutions, when we are unsuccessful with $\int dx \cdot \phi(x)$. Again since in each of the terms,

$\int \frac{dx \cdot \mathrm{R}(x)}{\mathrm{R}_1(x)} \cdot \phi_1(x)$, $\frac{\mathrm{R}(x)}{\mathrm{R}_2(x)}$ may be reduced to a series of terms of the form Ax^n and $\frac{A}{(x+a)^r}$; therefore the fluent depends on a series of terms $\int dx \cdot x^n \cdot \phi_1(x)$, and $\int dx \cdot (x+a)^{-n} \cdot \phi_1(x)$. In the latter case, the form of $\phi_1(x)$ is not changed by substituting x for $x + a$, and $\therefore$ the fluents of all irrationals are determinable by $\int dx \cdot x^{\pm n} \cdot \phi_1(x)$.

Cor. 1. If we multiply the denominator of

$$\frac{dx \cdot \mathrm{R}(x)}{\mathrm{R}_1(x) \cdot \sqrt{ax^2 + bx + c} + \mathrm{R}_2(x) \cdot \sqrt{\alpha x^2 + \beta x + \gamma}}$$

by its rationalizing factor, the fluxion will be reduced to two terms, which admit distinct rationalities.

Cor. 2. Sometimes by the substitution of a new variable, for some function of x, the fluxion will be divided into a series of terms, each of which may be separately made rational. But unfortunately no general principle has been discovered, to which these reductions can be referred.

Cor. 3. As the fluent of each term can sometimes be found

apart, when the fluent of the whole cannot be found at once; so conversely, the fluent of a series of terms may be found, when each separate term surpasses the powers of analysis. Thus we know

$$\int \frac{d\varphi(x) + d\varphi_{,}(x) + \ldots}{\varphi(x) + \varphi_{,}(x) + \ldots}$$

But we do not know

$$\int \frac{d\varphi(x)}{\varphi(x) + \varphi_{,}(x) + \ldots} + \int \frac{d\varphi_{,}(x)}{\varphi(x) + \varphi_{,}(x) + \ldots} + \ldots$$

Again, let $\varphi(x)$ be such a function of x,

that $\varphi'(x) = \frac{x}{e}$; let $\varphi_{,}(x) = x$;

Then $\int dx \,.\, \varphi(x) = x \,.\, \varphi(x) - \int d\varphi(x) \,.\, x$

$$= x \,.\, \varphi(x) - \int d\varphi(x) \,.\, \varphi'(x) \,.\, e$$

$$= x \,.\, x - e \int d x \,.\, \varphi(x)$$

$$\therefore \int dx \,.\, \varphi(x) + e \int d x \,.\, \varphi(x) = x \,.\, x$$

Which theorem admits farther extension, and may be applied to elliptic arches.

Should the above processes for rendering the fluxion rational fail us, we must attempt the fluxion at once in its irrational state, for which purpose I shall add a few miscellaneous observations.

1. If $\varphi(x)$, $\varphi_{,}(x)$ be any algebraic functions,

then $d\left\{\varphi(x) + \log. \varphi_{,}(x)\right\} = d\varphi(x) + \frac{d\varphi_{,}(x)}{\varphi_{,}(x)}$ is an algebraic expression. Whenever, therefore, we meet with an

algebraic fluxion, we may legitimately try $\phi(x) + \log. \underset{1}{\phi}(x)$, as a form to which the fluent may possibly belong.*

2. It presents three cases: 1st. where the fluent is wholly algebraic, for which we assume some expression so general, that its fluxion will include the given fluxion, if it admit an algebraic fluent; or we find the fluent implicitly by an equation: 2dly. where the fluent is mixed, when we attempt to separate the algebraic part: 3dly. where the fluent is purely logarithmic, when we assume, as in the first case, some expression with indeterminate constants, sufficiently general to include the given fluxion.

3. In assuming for an algebraic fluxion, it must be observed, that the fluent will always be a surd of the same order as the fluxion. On this principle WARING gives some assumptions for surds of the second order, but nothing has been attempted generally for surds of all orders, for want of some definite form which should include them all. In irrationals of the first order, the fluxion may always be reduced to series of terms, such as

$$dx \cdot \left(x + \underset{1}{a}\right)^{\alpha} \cdot \left(x + \underset{2}{a}\right)^{\beta} \cdots \left(x + \underset{n}{a}\right)$$

where the factors are all different, and where the indices are positive, negative, fractions, integers, or unity. Then let $\underset{n-1}{\mathrm{R}}(x)$ be any expression $cx^{n-1} + \underset{1}{c}x^{n-2} + \ldots$ with indeterminate coefficients. Assume for the fluent

$$\frac{\left(x + \underset{1}{a}\right)^{\alpha+1} \cdot \left(x + \underset{2}{a}\right)^{\beta+1} \cdots \left(x + \underset{n}{a}\right)^{\nu+1}}{\underset{n-1}{\mathrm{R}}(x)}$$

* It is obvious, that the fluent of an algebraic fluxion cannot be of the form $\phi(x) + \underset{1}{\phi}(x) \cdot \log. \underset{2}{\phi}(x)$, for its fluxion $d\phi(x) + d\underset{1}{\phi}(x) \cdot \log. \underset{2}{\phi}(x) + \underset{1}{\phi}x \cdot \frac{d\underset{2}{\phi}(x)}{\underset{2}{\phi}(x)}$ is a transcendent.

Its fluxion will be

$$\frac{\underset{n-1}{\mathrm{DR}}\,(x)\cdot\left(x+\overset{a}{\cdot}\right)\cdot\left(x+\overset{a}{\cdot}\right)\cdots\left(x+\overset{a}{\cdot}\right)}{\left\{\underset{n-1}{\mathrm{R}}\,(x)\right\}^{n}}$$

$$-\frac{\underset{n-1}{\mathrm{R}}\,(x)\cdot\left\{(\alpha+1)\cdot(x+\overset{a}{\cdot})\cdots(x+\overset{a}{\cdot})+(\beta+1)\cdot(x+\overset{a}{\cdot})\cdots(x+\overset{a}{\cdot})+\cdot\right\}}{\left\{\underset{n-1}{\mathrm{R}}\,x)\right\}^{2}}$$

multiplied by $dx\,.\,(x+\overset{a}{\cdot})^{\alpha}\ldots(x+\overset{a}{\cdot})^{\gamma}$ the original fluxion.

Now that the two expressions may be equal, the coefficient found above must be $=1$, or we must have

$$\underset{n-1}{\mathrm{DR}}\,(x)\,.\,\mathrm{Q}-\underset{n-1}{\mathrm{R}}\,(x)\,.\,\mathrm{Q}'=\left\{\underset{n-1}{\mathrm{R}}\,(x)\right\}^{n}$$

where Q and Q′ are the expressions in the coefficient involving a, a, a. By equating the terms in this equation, the indeterminates c, c, c, &c. may be found; but the reduction will often be impossible, as there are more equations to be satisfied than there are indeterminates.

4. If any index $\alpha, \beta, \gamma = -1$, the expression fails, and there is no algebraic fluent; also WARING says, that if the dimensions of a fluxional coefficient be $= -1$, the fluent cannot be algebraic.

5. If $\varphi(x)$ be an irrational function, let $z = \int dx\,.\,\varphi(x) = \int dx\,.\,v$; then since $\mathrm{R}\{x, v\} = o$, there are cases, where we can determine, $\mathrm{R}\{x, z\} = o$.*

6. If $\varphi(x), \pi(x)$ be irrational functions of x, we have

$$\int dx\,.\,\varphi(x) = \int dx\,.\,\{\varphi(x)+\pi(x)\} - \int dx\,.\,\pi(x)$$

Now let $\pi(x)$ be so assumed, that

* See Phil. Trans. 1764.—EMERSON's Fluxions.

$\int dx \cdot \{\phi(x) + \pi(x)\} = \underset{1}{\phi}(x)$, and we have

$$\int dx \cdot \phi(x) = \underset{1}{\phi}(x) - \int dx \cdot \pi(x).$$

If therefore $\pi(x)$ be a simpler expression than $\phi(x)$, the fluxion will be reduced to a simpler form. In order to find $\pi(x)$; $\underset{1}{\phi}(x)$ is assumed with indeterminate coefficients, so that its fluxion may be of the same form as $dx \cdot \phi(x)$. Now equate the similar terms in the two expressions, and the indeterminates may be found. But as there may be more equations than indeterminates, we add $\pi(x)$ a function of the same form, and containing indeterminates of sufficient number, to satisfy all the deficient equations. We shall thus have $D\underset{1}{\phi}(x) = \phi(x) + \pi(x)$ and $\therefore \underset{1}{\phi}(x) = \int dx \cdot \{\phi(x) + \pi(x)\}$ by which the difficulty may be reduced to finding $\int dx \cdot \pi(x)$. Reductions of particular kinds were discovered by SIMPSON and others, but this is universally applicable.

7. It may be of advantage to reduce the index of the variable under the radical, which may sometimes be effected. In $dx \cdot \{x^{m+n} + 1\}^{\frac{1}{n}}$, assume $x^{m+n} + 1 = v^n$;

Then we have

$$dx \cdot \{x^{m+n} + 1\}^{\frac{1}{n}} = \frac{n}{m+n} \cdot dv \cdot v^n \cdot (v^n - 1)^{\frac{1}{m+n} - 1}$$

And in the same manner surds of one order may be transformed into another.

8. If the fluent be wholly logarithmic, we may assume for irrationals of the first order

$$\log. \{\underset{0}{R}(x) + \underset{1}{R}(x) \cdot (x+a)^{\alpha} \cdot (x+b)^{\beta} \ldots + \underset{2}{R}(x) \cdot (x+c)^{\gamma} \ldots + \}$$

9. I shall conclude by observing that the fluxion may always be made rational, if the fluent be wholly algebraic, or wholly logarithmic. Thus, if $\phi(x)$ be any algebraic function, take $x = \phi^{-1} R(v)$,

Then $d\phi(x) = d\left(\phi\phi^{-1} R(v)\right) = d R(v)$

and $d\left(\log. \phi(x)\right) = d\left(\log. \phi\phi^{-1} R(v)\right) = \frac{d R(v)}{R(v)}$

are both rational. If the fluent be of the mixed form $\phi(x) + \log. \phi_{,}(x)$, its fluxion may be made rational, if $R, R_{,}$ can be so assumed that $\phi^{-1} R(x) = \phi_{,}^{-1} R_{,}(x)$; and it may always be effected by introducing two new variables.

First let $x = \phi^{-1} R(v)$, and the fluxion becomes $d R(v) + \frac{d\left(\phi\phi^{-1} R(v)\right)}{\phi\phi^{-1} R(v)}$; now let $v = R^{-1} \phi\phi^{-1} R_{,}(z)$, and we get $d R(v) + \frac{d R_{,}(z)}{R_{,}(z)}$ which is wholly rational.

EDWARD FRENCH BROMHEAD.

Note.—As modern analysts have in general confounded the fluxions, either with the increments or the derived functions, it may not be superfluous to state precisely, what is meant by the symbols d and D.

If it be possible, (which must be shown in each particular case) to expand $\phi(x+v)$ in the form $\phi(x) + \phi_{,}(x) \cdot + \phi_{,}(x) \cdot v^{n} + \ldots$; then $\phi_{,}(x)$ is called the derived function of $\phi(x)$, and its relation to $\phi(x)$ is thus expressed, $\phi_{,}(x) = D\phi(x)$. Hence, if x be considered a function of itself, we have $(x + v) = (x) + D(x) \cdot v$, and $\therefore$ $D(x) = 1$.

Now to avoid a constant reference to the variable x, of which other variables are considered as functions, we introduce fluxions. If $y, z, w, \ldots$ are functions of the

same variable, then dy, dx, dw, . . are expressions *proportional* to the derived functions of y, x, w, . . whatever may be the variable of which they are common functions. Hence $\frac{dy}{dx} = \frac{Dy}{Dx}$; and if y be a function of x, or $= \varphi(x)$, then $\frac{dy}{dx} = \frac{D\varphi(x)}{Dx} = D\varphi(x)$ and $\therefore dy = dx \,.\, D\varphi(x)$.

Moreover, since the derived functions are in the limiting ratio of the increments, so also are the fluxions. From this consideration we can in the applications of analysis, *practically* determine the ratio of the fluxions, when the derived functions are unknown.

ERRATA.

Page 72, line 20, for *parts*, read *part*.
—— 73, line 3, for *between*, read *below*.
—— 98, line 4 from bottom, dele the comma after A.
—— 101, line 6 from bottom, dele BH.
—— 102, line 4, for *axes*, read *axis*.
—— 164, line 11, dele the comma between m and n.
—— 174, line 7, for *consisted of*, read *consisted in*.
—— ——, line last, for m, n, read m, m.
—— 191, line 13, for $\varphi\varphi^{-1}x$, read $\varphi^{-1}\varphi x$.
—— 213, line 14, for $\psi\downarrow(x, y)$, read $\varphi'\downarrow(x, y)$.
—— 214, line 10, dele "*in an infinite number of ways*".
—— 224, line 22, for $f(a)$, read $f(x)$.
—— 226, line 24, for $= x$, read $= u$.
—— 232, line 16, *in the denominator*, for $1-$, read $1+$.
—— ——, line 18, *ditto*, *ditto*, for $1-$, *read* $1\pm$.
—— 251, line 9, for $\frac{d\downarrow\left(x, \frac{1}{y}\right)}{dx}$ read $\frac{d\downarrow\left(x, \frac{1}{y}\right)}{dx}$
—— ——, line 11, for *d in both numerator*, read d^n.
—— —— line 13, for $\left(\frac{x}{y}\right)$ read $x\,\varphi\left(\frac{x}{y}\right)$.

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	The Philosophical Magazine for March.	Mr. A. Tilloch.

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	The European Magazine for March.	The Editor.
	A Chinese Poem inscribed on Porcelain, in the 33d year of the Cycle, A. D. 1776, with a double Translation and Notes, by S. Weston, F. R. S. and F. L. S. London, 1816. 12°	Rev. Stephen Weston.
25.	Journal des Mines, No. 161, 168, 214.	Ecole des Mines.
	Essays, Religious and Moral. London, 1812, 12°	Isaac Hawkins Brown, Esq
	The Principles of Fluxions, by the Rev. William Dealtry. London, 1816. 8°	Rev. William Dealtry.
	The First Centenary of a Series of concise and useful Tables of all the complete decimal Quotients which can arise from dividing a Unit or any whole Number less than each Divisor by all Integers from 1 to 1024.	Henry Goodwyn, Esq.
	Des Beautés poetiques de toutes les Langues considerés sous le Rapport de l'Accent et du Rythme, par l'Abbé Ant. Scoppa. Paris, 1816. 8°	The Abbé Scoppa.
	Plants of the Coast of Coromandel, Vol. III. No. II. fol.	The Committee of Warehouses of the East India Company.
	Mémoires de la Classe des Sciences mathematiques et physiques de l'Institut Imperial de France for 1810, 1811, and Part 1, for 1812. Paris. 4°	The Royal Institute of France.
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May 2.	Annals of Philosophy, No. 41.	Dr. Thomas Thomson.
	The European Magazine for April.	The Editor.
	The Monthly Review for April.	Mr. G. E. Griffiths.
	The Philosophical Magazine for April.	Mr. Alexander Tilloch.
9.	The Principles of Population and Production, by John Weyland, Jun. Esq. London, 1816. 8°	John Weyland, Esq.
16.	Catalogue raisonné des Diamants dans le Cabinet de Sir Abraham Hume, Bart. par M. Le Comte de Bournon. Londres, 1815.	Sir Abraham Hume, Bart.
30.	Annals of Philosophy, No. 42.	Dr. Thomas Thomson.
June 13.	Journal des Mines, No. 219 and 220.	Ecole des Mines.
	De l'Education physique de l'Homme par M. Friedlander, D. M. Paris, 1815. 8°	Mr. Friedlander.
	Von der Natur und den Eigenschaften des Lichtes von Placidus Heinrich, St. Petersburgh, 1808. 4°	M. Placidus Heinrich.
	Die Phosphorescenz der Korper oder die im Dunkeln bemerkbaren Lichtphanomene der anorganischen Natur, von Placidus Heinrich. Nurnberg, 1815. 4°	

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	Memoirs of the American Academy of Arts and Sciences, Vol. III. Part 2.	American Academy of Sciences.
	Transactions of the Horticultural Society, Vol. II. Part 3 and 4.	The Horticultural Society.
	Astronomical Observations made at the Radcliffe Observatory at Oxford, from May the 1st. 1815, to May the 1st. 1816. MS.	Dr. Robertson, by the direction of the Trustees of Dr. Radcliffe.
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	The Philosophical Magazine for May.	Mr. Alexander Tilloch.
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July 4.	Leçons de Mechanique analytique, Tome II. par M. Prony.	M. Prony.
	A Journal of Science and the Arts, No. 2.	Mr. Murray.
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	Monthly Review for June.	Mr. G. E. Griffiths.
	European Magazine for June.	The Editor.
	An Engraving of fossil Bones from Cephalonia.	Count Valsamachi.
	An Address on Zoology and Botany, by C. S. Rafinesque.	C. S. Rafinesque, Esq.
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